This question paper contains 4 printed pages]

Roll No. 2 2 0 6 9 5 6 3 0 0 2

S. No. of Question Paper : 1121

Unique Paper Code : 2352013503

Name of the Paper : Partial Differential Equations

Name of the Course : Bachelor of Science (Honours Course) Mathematics

Semester : V

Duration : 3 Hours

Maximum Marks : 90

*(Write your Roll No. on the top immediately on receipt of this question paper.)*

*All* the six questions are compulsory.

Attempt any *two* parts from each question.

Use of calculator is not allowed.

1. (*a*) Define the linear and non-linear partial differential equations with examples. Find the first order partial differential equation satisfied by the family of right circular cones whose axes coincide with the $z$-axis, and is given by :

$$x^2 + y^2 = (z - c)^2 \tan^2\alpha.$$

(*b*) Solve the following initial value problem using method of characteristics :

$$u_t + 2uu_x = v - x, \; v_t - cv_x = 0 \text{ with } u(x, 0) = x, \; v(x, 0) = x.$$

(*c*) Solve the equation $u_x + xu_y = y$ with the Cauchy data $u(1, y) = 2y$.

2. (a) Reduce the following equation into canonical form and then find the general solution :

$$u_x + 2xyu_y = x.$$

(b) Apply the method of separation of variables $u(x, y) = f(x)\, g(y)$ to solve the following equation :

$$u_x + 2u_y = 0, \quad u(0, y) = 3e^{-2y}.$$

(c) Find a complete integral of the equation by using Charpit's method :

$$p = (z + qy)^2.$$

3. (a) Show that the equation of motion of the vibrating string is :

$$u_{tt} = c^2 u_{xx}, \quad c^2 = \mathrm{T}/\rho,$$

where T is the tension at the end point of the string and $\rho$ is the density.

(b) Classify and determine the region in which the following equation is hyperbolic, parabolic, or elliptic, and transform the equation in the respective region to canonical form :

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0.$$

(c) Find the traffic density $\rho(x, t)$, satisfying :

$$\frac{\partial \rho}{\partial t} + x\sin(t)\,\frac{\partial \rho}{\partial x} = 0,$$

with the initial condition $\rho_0(x) = 1 + \dfrac{1}{1+x^2}$.

4. (a) Transform the following equation to the form $u_{\xi\eta} = cv$, $c$ = constant,

$$3u_{xx} + 7u_{xy} + 2u_{yy} + u_y + u = 0,$$

by introducing the new variables $v = ue^{-(a\xi + b\eta)}$, where $a$ and $b$ are undetermined coefficients.

(b) Define the homogeneous and non-homogeneous of partial differential equations with examples, and find the general solution of the equation :

$$u_{xxxx} - u_{yyyy} = 0.$$

(c) Find the general solution of the equation :

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y + e^{2x+y}.$$

5. (a) Determine the solution of the initial-value problem for the semi-infinite string with a fixed end :

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, && 0 < x < \infty, t > 0 \\ &u(x, 0) = f(x), u_t(x, 0) = g(x), && 0 \le x < \infty \\ &u(0, t) = 0, && 0 \le t < \infty \end{aligned}$$

(b) Determine the solution of the initial-value problem for the semi-infinite string with free end :

$$\begin{aligned} &u_{tt} = 9u_{xx}, && 0 < x < \infty, t > 0 \\ &u(x, 0) = 0, u_t(x, 0) = x^3, && 0 \le x < \infty \\ &u_x(0, t) = 0, && 0 \le t < \infty \end{aligned}$$

(c) Determine the solution of the initial-value problem with non-homogeneous boundary conditions :

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, && 0 < x < \infty, t > 0 \\ &u(x, 0) = \sin x, u_t(x, 0) = x^2, && 0 \le x < \infty \\ &u(0, t) = x, && 0 \le t < \infty \end{aligned}$$

6. (*a*) Determine the solution of the Cauchy problem for non-homogeneous wave equation :

$$u_{tt} - c^2 u_{xx} - \sin x = 0, \quad u(x, 0) = \cos x, \quad u_t(x, 0) = 1 + x.$$

(*b*) Determine the solution of the initial-value problem :

$$\begin{aligned} &u_{tt} = 16u_{xx}, && 0 < x < \infty, t > 0 \\ &u(x, 0) = 1 + x, u_t(x, 0) = x^3, && 0 \le x < \infty \\ &u_x(0, t) = \cos x, && 0 \le t < \infty \end{aligned}$$

(*c*) Determine the solution of the initial-value problem :

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0 && 0 < x < \infty, t > 0 \\ &u(x, 0) = \log(1 + x^2), u_t(x, 0) = 2, && 0 \le x < \infty \end{aligned}$$