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Your Roll No. ²²⁰⁶⁹⁵⁶³⁰⁰².....

Sr. No. of Question Paper : 1102 I

Unique Paper Code : 2352013502

Name of the Paper : Ring Theory

Name of the Course : B.Sc. (H) Mathematics

Semester : V - DSC-14

Duration : 3 Hours

Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All questions are compulsory and are of 15 marks each.
3. Attempt any six parts from Question 1. Each part is of 2.5 marks.
4. Attempt any two parts from each of the Questions 2 to 6. Each part is of 7.5 marks.
5. Use of calculator is not allowed.

P.T.O.

1.
 - (i) Find all the units of $\mathbb{Z}_7[x]$.
 - (ii) Check whether $\mathbb{Q} \oplus \mathbb{Q}$ is an integral domain or not.
 - (iii) Give an example of a subring S of a ring R which is not an ideal of R .
 - (iv) Prove that a ring homomorphism carries an idempotent to an idempotent.
 - (v) Let ϕ be a ring homomorphism from a ring R to a ring S . If R has unity 1 , $S \neq \{0\}$ and ϕ is onto then prove that $\phi(1)$ is the unity of S .
 - (vi) Let $f(x) = 2x^5 + 14x^2 - 21x + 7$. Is $f(x)$ an irreducible polynomial over \mathbb{Q} ? Justify your answer.
 - (vii) Let D be an integral domain. Suppose that $p, q \in D$ and $q \neq 0$. Show that if p is not a unit, then $\langle pq \rangle$ is a proper subset of $\langle q \rangle$.
 - (viii) Explain why $3x^2 + 6$ is reducible over \mathbb{Z} .
2.
 - (a) Prove that intersection of two subrings in a ring R is a subring of R . Is the union of two subrings necessarily a subring of R ? Justify your answer.
 - (b) Find all the units, zero divisors and idempotent elements in $\mathbb{Z}_3 \oplus \mathbb{Z}_6$.

(c) Prove that \mathbb{Z}_n , the ring of integers modulo n , is a field if and only if n is a prime.

3. (a) Let R be a commutative ring with unity and let $U(R)$ denote the set of units of R . Prove that $U(R)$ is a group under multiplication. Also, find $U(\mathbb{Z}[i])$.

(b) Define the characteristic of a ring. Prove that the characteristic of an integral domain is either 0 or prime.

(c) Prove that in a commutative ring R with unity, an ideal A is a maximal ideal if and only if $\frac{R}{A}$ is a field.

4. (a) Prove that the ideal $\langle x \rangle$ is a prime ideal in $\mathbb{Z}[x]$ but not a maximal ideal in $\mathbb{Z}[x]$.

(b) Let ϕ be a ring homomorphism from a ring R onto a ring S . Prove that $\frac{R}{\text{Ker } \phi} \approx S$.

(c) Determine all ring homomorphisms from $\mathbb{Z}_4 \rightarrow \mathbb{Z}_{10}$.

5. (a) Let F be a field and let $I = \{a_0 + a_1x + \dots + a_nx^n : a_0, a_1, \dots, a_n \in F \text{ and } a_0 + a_1 + \dots + a_n = 0\}$. Show that I is an ideal of $F[x]$ and find a generator for I .

- (b) Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1 \in \mathbb{Z}_7[x]$. Determine the quotient and remainder obtained when $f(x)$ is divided by $g(x)$.
- (c) Prove that the product of two primitive polynomials is a primitive polynomial.
6. (a) Show that $p(x) = x^3 + x + 1$ is an irreducible polynomial over \mathbb{Z}_2 .

Let $M = \langle x^3 + x + 1 \rangle$ be an ideal of $\mathbb{Z}_2[x]$.

Show that $F = \frac{\mathbb{Z}_2[x]}{M}$ is a field of order 8. Exhibit

all the 8 elements of F . Find the product of $x^2 + x + 1 + M$ and $x^2 + 1 + M$ and express it as a member of F .

- (b) In a principal ideal domain, prove that the element is irreducible if and only if it is prime.
- (c) Show that integral domain $\mathbb{Z}[t]$ is Euclidean Domain. Is $\mathbb{Z}[i]$ a Unique Factorization Domain? Justify.