BMH203:Real Analysis

Answer the following Multiple Choice Questions

1. A set is said to be finite if

   (a) it is empty.

   (b) it has certain number of elements.

   (c) either a) or b).

   (d) none of the above.

   Ans- c)

2. If S,T are sets such that S is a finite set and $T \subseteq S$, then which of thefollowing holds

   (a) T is infinite.

   (b) T is empty.

   (c) T is also finite.

   (d) cannot say.

   Ans- c)

3. A denumerable set is

   (a) Countable set.

   (b) Uncountable set.

   (c) Empty set.

   (d) none of these.

   Ans- a)

4. Any bounded set S can be

   (a) bounded above.

   (b) bounded below.

   (c) both a) and b).

   (d) all of the above.

   Ans- d)

5. What is/are the limit point(s) of $\mathbf{Z}$?

   (a) 0

   (b) Infinite number of points.

(c) No point.

(d) 1

Ans- c)

6. If all the subsequences of any sequence converge to L, then

(a) the sequence will also converge to L.

(b) the sequence may or may not converge to L.

(c) the sequence will not converge to L.

(d) none of the above.

Ans- b)

7. The given sequence

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

(a) will diverge.

(b) will converge.

(c) will neither converges nor diverges.

(d) cannot say.

Ans- b)

8.

$$\sum \frac{1}{n^p}$$

will converge if

(a) $-\infty \leq p \leq \infty$

(b) $0 \leq p$

(c) $p > 1$

(d) $p = 1$

Ans- c)

9. If the sequence is convergent then the subsequence

(a) will also converge.

(b) will diverge.

(c) cannot say.

(d) non of these.

Ans- a)

10. A sequence is diverges if
    (a) that sequence has two divergent subsequences whose limits are notequal.
    (b) that sequence is unbounded.
    (c) cannot be determined.
    (d) either a) or b).

    Ans- d)

Answer the following questions

Questions

1. Prove that set **N** of natural numbers is an infinite set.
2. If $z, c \in \mathbf{R}$, such that $z + a = a$ then show that $z = 0$.
3. Show that there is no $r \in \mathbf{R}$ such that $r^2 = 2$.
4. If $a, b \in \mathbf{R}$ such that $a \geq 0, b \geq 0$ then show that

$$a < b \Leftrightarrow a^2 < b^2.$$

5. If a and b are positive real numbers then find the arithmetic-geometricinequality.
6. Prove that supremum of a set is unique.
7. Let S be a non-empty, bounded subset of **R** and a R. Prove that $Sup(a + S) = a + sup(S)$.
8. Prove that set **R** of real numbers is uncountable.
9. Find $\lim_{n\to\infty} \frac{3n+1}{2n+}$.
10. If $c > 0$ then show that $\lim c^{\frac{1}{n}} = 1$.
11. Find the value of

$$\lim \frac{2n^2 + 3}{2}$$

$\lim_{n\to\infty}$ $3n + 5$

12. If (X_n) is a convergent sequence with $a \leq X_n \leq b$. Prove that $a \leq \lim(X_n) \leq b$.

13. Find the limit of sequence $(n^{\frac{1}{n^2}})$.

14. Find the limit of sequence $(n!)^{\frac{1}{n^2}}$.

15. Establish the convergence of $(\frac{1}{\sqrt{n}})$ and find the limit.

16. Establish the convergence or divergence of the sequence (x_n) where

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

17. Test the convergence of the sequence (a_n) defined by

$$a_1 = \frac{3}{2}, \quad a_{n+1} = 2 - \frac{1}{a_n} \text{ forall } n \geq 1.$$

18. Prove that every Cauchy sequence is bounded.

19. Test the convergence of the series $\sum r^n$.

20. Test the convergence of the series $\sum e^{-n^2}$.

Answers

1. Assume that $\mathbf{N}$ is a finite set.
   ⇒ there exists a bijection $f : \mathbf{N_n} \to \mathbf{N}$ for some $n \in \mathbf{N}$.
   ⇒ $f^{-1} : N \to N_n$ also a bijection.
   Consider N_{n+1}

$$h : N_{n+1} \to N_n$$
$$h(x) = g(x) \text{ for all } x \in N_{n+1}$$

If h is one-one function ,then

$$\begin{aligned} h(x_1) &= h(x_2) \\ \Rightarrow g(x_1) &= g(x_2) \\ \Rightarrow x_1 &= x_2 (\textit{since is a one – one function}) \end{aligned}$$

⇒ h is one-one function. But, this is contradiction to *Pigenhole Principal*. So our assumption is wrong and N is not a finite set.

Hence, N is an infinite set.

2.

$$\begin{aligned} z &= z + 0 ----(existence\ of\ identity\ element) \\ &= z + (a + (-a)) ----(existence\ of\ inverse\ element) \\ &= (z + a) + (-a) ----(Associativity) \\ &= a + (-a) ----(z + a = 0 \to given) \\ z &= 0 ----(inverse\ property) \end{aligned}$$

Hence, $z = 0$ proved.

3. On the contrary lets assume that there exists a rational number $\frac{p}{q}$ such that $\left(\frac{p}{q}\right)^2 = 2$ $gcd(p, q) = 1$

$$\begin{aligned} \frac{p^2}{q^2} &= 2 \\ \Rightarrow p^2 &= 2q ----(1) \end{aligned}$$

Which is even $\Rightarrow p^2$ is even.
$\Rightarrow p$ is even.
Let $p = 2m$ for some $m \in N$. From (1) we get

$$\begin{aligned} (2m)^2 &= 2q^2 \\ 4m^2 &= 2q^2 \\ \Rightarrow q^2 &= 2m \end{aligned}$$

$\Rightarrow q^2$ is even.
So q is even.
Since, p and q are even the $gcd(p, q) = 1$.
We get a contradiction and thus,our assumption is wrong.Hence,there is no rational number r such that $r^2 = 2$.

4. **CASE 1** If $b = 0$ and $a \leq b$ (given)

$$\begin{aligned} \Rightarrow 0 \leq a \leq b &= 0 \\ \Rightarrow a &= 0 \end{aligned}$$

$$\Rightarrow a^2 = 0 = b^2$$

CASE 2 If $b \neq 0$, $b > 0 \Rightarrow b^2 > 0$
and $a = 0 \Rightarrow a^2 = 0$
$\Rightarrow a^2 < b^2$

CASE 3 If $a \neq 0, b \neq 0$ and $a = b$
Consider,

$$b^2 - a^2 = (b - a)(b + a)$$

$$\Rightarrow a > 0, b > 0$$

$$\Rightarrow a + b > 0 (a < b \text{ given})$$

$$\Rightarrow b - a > 0$$

$$\Rightarrow (b - a)(b + a) > 0$$

$$\Leftrightarrow b^2 - a^2 > 0$$

$$\Leftrightarrow a^2 < b^2$$

If

$$b^2 - a^2 > 0$$

$\Rightarrow (b-a)(b+a) > 0$ ----(since both $(b-a)$ and $(b+a)$ are >0)

$\Rightarrow b - a > 0$

$\Rightarrow b > a$

Hence, $a < b \Leftrightarrow a^2 < b^2$

5. **CASE 1** $a \neq b, a > 0, b > 0$
$\Rightarrow \sqrt{a} > 0$ and $\sqrt{b} > 0$
Consider $(\sqrt{a} - \sqrt{b})^2 > 0$
$\Rightarrow a - 2\sqrt{ab} + b > 0$
$\Rightarrow a + b > 2\sqrt{ab}$
$\Rightarrow \sqrt{ab} < \frac{a+b}{2}$
CASE 2 $a = b, a > 0, b > 0$
LHS $= \sqrt{ab} = \sqrt{aa} = \sqrt{a^2} = a$

RHS $= \frac{a+b}{2} = \frac{a+a}{2} = \frac{2a}{2} = a$

Hence,
LHS=RHS
$\Rightarrow \sqrt{ab} = \frac{a+b}{2}$
So, from CASE 1 and CASE 2 we get

$$\sqrt{ab} \leq \frac{a+b}{2}$$

6. If possible let u_1 *and* u_2 be two supremum of S.i.e.$u_1 = Sup(S)$ and $u_2 = Sup(S)$
$u_1 < u_2$ or $u_2 < u_1$ or $u_1 = u_2$
CASE 1 If $u_1 < u_2$
$u_2 = Sup(S) \Rightarrow u_2$ is least upper bound. since $u_1 < u_2 \Rightarrow u_1$ is not an upper bound.
$\Rightarrow u_2 = Sup(S)$
Which is a contradiction
$\Rightarrow u_1$ is not less than u_2.
CASE 2 If $u_2 < u_1$
$u_1 = Sup(S) \Rightarrow u_1$ is least upper

bound. since $u_2 < u_1 \Rightarrow u_2$ is not an
upper bound.
$\Rightarrow u_2 = Sup(S)$
Which is a contradiction
$\Rightarrow u_2$ is not less than
u_2. So, we must have
$u_1 = u_2$so, $\Rightarrow Sup(S)$ is
unique.

7. Let $u = Sup(S)$
We have to show that $Sup(a + S) = a$
$+ uu = Sup(S) \Rightarrow x \leq u$ for all $x \in S$
$\Rightarrow a + x \leq a + u$ for all $x \in S$
$\Rightarrow a + u$ is an upper bound of $a + S$
$\Rightarrow Sup(a + S) \leq a + u$—— (1)

Let v be any arbitrary upper bound of
$\Rightarrow a + S \bar{a} + x \leq v$ for all $x \in S$
$\Rightarrow x \leq v - a$ for all $x \in S$
$v - a$ is an upper bound
of $S. v - a \geq Sup(S) = u$
$\Rightarrow v - a \geq u$
$\Rightarrow v \geq a + u$
$\Rightarrow Sup(a + S) \geq a + u$——(2) (since $Sup(a + S)$ ia also an upper bound)
(1) and (2) both are possible only if

$$\Rightarrow Sup(a + S) = a + u$$

$$\Rightarrow Sup(a + S) = a + Sup(S)$$

8. To prove set of **R** uncountable we first prove that set $I = [0, 1]$ is uncount-able.
Let I be countable and let $I = \{x_1, x_2, x_3, \ldots\}$ be the enumeration of I.Choose, $I_1 \supseteq I$ such that $x_1 \notin I_1$
Now choose $I_2 \supseteq I$ such that $x_2 \notin I_3$
and $I_3 \subseteq I_2$ such that $x_3 \notin I_3$.

So, we obtain a nested sequence of close intervals ,i.e.

$$I_1 \supseteq I_2 \supseteq I_3 \ldots \supseteq I_n \supseteq I_{n+1} \ldots$$

By nested interval property there exists a $\chi \in I$ such that $\chi \quad I_n$ for all
$n \in N$.
But this isn't happening
$\Rightarrow \chi \notin x_n$ for all $n \in N$.
The enumeration of I is not a complete listing if the elements of I.
$\Rightarrow I$ is not

countableand
since $I \subseteq \mathbf{R}$
$\Rightarrow \mathbf{R}$ is not countable.

9. Let $\mathrm{E} >$ 0;

$$x_n = \frac{3n + 1}{2n + 5}$$

Consider,
$$\frac{3n+1}{2n+5} - \frac{3}{2} = \frac{-13}{4n+10}$$

$$= \frac{13}{4n+10} < \frac{13}{n} < \text{E}$$

if $n > \frac{13}{E}$

Choose, $k \in \mathbf{N}$ such that $k > \frac{13}{E}$

if $n \geq k$,

then
$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \text{E}$$

$$\Rightarrow \lim \frac{3n+1}{2n+} = \frac{3}{2}$$

10. **CASE 1** $c = 1$
$c^{\frac{1}{n}} \to 1^{\frac{1}{n}} = 1, 1, 1, 1, \ldots$
$(c^{\frac{1}{n}}) \to (1, 1, 1, 1, \ldots) \to 1$
CASE 2 $c > 1$
$c^{\frac{1}{n}} > 1$

$$\Rightarrow c^{\frac{1}{n}} = 1 + d_n - - - - - (1)(where\, d_n > 0)$$

$$\Rightarrow c = (1 + d_n)^n$$

$$\Rightarrow c = (1 + nd_n) + {}^nc_2(d_n) + \ldots - - - - - (Binomial\ theorem)$$

All terms > 0

$$\Rightarrow c \geq 1 + nd_n$$

$$\Rightarrow d_n \leq \frac{c-1}{n}$$

From (1)
$$c^{\frac{1}{n}} - 1 = d_n \leq \frac{c-2}{n}$$

Let E > 0
$c^{\frac{1}{n}} - 1 \leq \frac{c-1}{n} <$ E if $n > \frac{c-1}{E}$

Let $k \in \mathbf{N}$ such that $k > \frac{n}{E}$

if $n \geq k$, then $\left| c^{\frac{1}{n}} - 1 \right| < E$

$\Rightarrow \lim(c^{\frac{1}{n}}) = 1$

CASE 3 $0 < c < 1$; for some $c^{\frac{1}{n}} = \frac{1}{1+h_n}$ for some $h_n > 0$.

$$\Rightarrow c = \frac{1}{(1 + h_n)^n} \leq \frac{1}{1 + nh_n} < \frac{1}{nh_n}$$

$$\Rightarrow 0 < h_n < \frac{1}{cn}$$

Consider,

$$0 < 1 - c^{\frac{1}{n}} = 1 - \frac{1}{1 + h_n} = \frac{h_n}{1 + h_n} < h_n$$

$$= \frac{h_n}{1 + h_n} < h_n < \frac{1}{cn}$$

$$\Rightarrow \left| 1 - c^{\frac{1}{n}} \right| < \frac{1}{cn} \Rightarrow \left| c^{\frac{1}{n}} - 1 \right| < \frac{1}{cn} < E$$

if $n > \frac{1}{cE}$

$\Rightarrow \lim c^{\frac{1}{n}} = 1$

11.

$$\lim_{n\to\infty} \frac{2n^2 + 3}{3n^2 + 5} = \lim_{n\to\infty} \frac{2 + \frac{3}{n^2}}{3 + \frac{5}{n^2}}$$

$$\frac{\lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{3}{n^2}}{\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{5}{n^2}}$$

$$\frac{2 + 0}{3 + 0} = \frac{2}{3}$$

12. Let (y_n) be a constant sequence with $y_n = a$ for all n
$(y_n) = (a, a, a, a, \dots) \to a$
$\Rightarrow y_n \leq x_n$ for all n
$\Rightarrow \lim(y_n) \leq \lim(x_n)$ (From theorem)
$\Rightarrow a \leq \lim(x_n)$—-(1)

Let, (z_n) where, $z_n = b$ for all
n$(z_n) = (b, b, b, b, \dots) \to b$
$\Rightarrow x_n \leq z_n$ for all n
$\Rightarrow \lim(x_n) \leq \lim(z_n)$
$\Rightarrow \lim(x_n) \leq b$—-(2)

From (1) and (2), we
get

$$a \leq \lim(x_n) \leq b$$

13. $y_n = (n^{\frac{1}{n^2}}$
when $n = 1; y_n = 1$(equal to 1)
when $n = 2; y_n = 2^{\frac{1}{2^2}} = 2^{\frac{1}{4}}$ (which is greater than 1)
as we go further we see all the elements of (y_n) are ≥ 1

$$1^n \leq y_n = n^{\frac{1}{n^2}}$$

$$n^{\frac{1}{n^2}} = (n^{\frac{1}{n}})^{\frac{1}{n}} \leq n^{\frac{1}{n}}$$

$1^n \leq n^{\frac{1}{n^2}} \leq n^{\frac{1}{n}}$ We know that $1^n \to 1$
$n^{\frac{1}{n}} \to 1$
So, by using Squeeze theorem we get,

$$\lim(n^{\frac{1}{n^2}}) = 1$$

14.

$$y_n = [(n!)^{\frac{1}{n^2}}]$$

Clearly, $\leq (n!)^{\frac{1}{n^2}}$
1^n

$$\begin{aligned} (y_n) &= (1, (2!)^{\frac{1}{2^2}}, (3!)^{\frac{1}{3^2}}, \dots) \\ n! &= n \cdot (n-1) \cdot (n-2) \cdot (n-3) \dots \cdot 1 \\ &\leq n \cdot n \cdot n \dots n = n^n \end{aligned}$$

$$n! \le n^n$$

$$\Rightarrow (n!)^{\frac{1}{n^2}} \le (n^n)^{\frac{1}{n^2}} = n^{\frac{1}{n}}$$

Since, $1^n \to 1$ and $n^{\frac{1}{n}}$
So, by squeeze theorem

$$\lim[(n!)^{\frac{1}{n^2}}] = 1$$

15. Let $x_n = \frac{1}{\sqrt{n}}$

$$n + 1 > n \Rightarrow \sqrt{n+1} > \sqrt{n}$$
$$\Rightarrow \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$
$$\Rightarrow x_{n+1} < x_n$$

$\Rightarrow (x_n)$ is decreasing

Clearly, $\frac{1}{\sqrt{n}} > 0$ for all n,i.e.,$x_n > 0$ for all .

$\Rightarrow (x_n)$ is bounded below.

$\Rightarrow (x_n)$ is convergent and

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = inf \left\{\frac{1}{\sqrt{n}}; n \in \mathbf{N}\right\} = 0$$

16.

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$
$$x_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1}$$

$$x_{n+1} = x_n + \frac{1}{n+1}$$
$$\Rightarrow x_{n+1} - x_n = \frac{1}{n+1} > 0$$
$$\Rightarrow x_{n+1} > x_n for\ all\ n$$

$\Rightarrow (x_n)$ is increasing.

$$x_{2^n} = 1 + \frac{1}{2} \quad \frac{1}{} \quad + \frac{1}{3} \quad + \frac{1}{} \quad + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \quad + \cdots \quad \frac{1}{2^{n-1}+1} + \cdots + \frac{1}{}$$

$$> \quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots + \frac{1}{2} + \cdots + \frac{1}{2}$$

$$= \quad 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \quad --- (n \text{ terms})$$

$$= \quad 1 + \frac{n}{2}$$

$$\Rightarrow x_{2^n} \quad > \quad 1 + \frac{n}{2} \text{ for all } n$$

As n increases; x_{2^n} also increases.

$\Rightarrow (x_n)$ is unbounded sequence.
$\Rightarrow x_n$ is divergent sequence.

17.

$$a_1 = \frac{3}{2}, a_2 = 2 - \frac{1}{a_1} = \frac{4}{3}$$

$$a_3 = 2 - \frac{3}{4} = \frac{5}{4}, a_4 = 2 - \frac{4}{5} = \frac{6}{5}$$

Clearly, $a_1 > a_2 > a_3 \ldots$

$. a_2 < a_1$

Let, P(n):$a_{n+1} < a_n$

for $n = 1$,P(1):$a_2 < a_1$,which is true.

Let P(k) be true,P(k):$a_{k+1} < a_k$

RTP P(k+1) is also true. P(k+1):$a_{k+2} < a_{k+1}$ Since,

$$a_{k+1} \quad < \quad a_k$$

$$\Rightarrow -a_{k+1} \quad > \quad -a_k$$

$$\Rightarrow -\frac{1}{a_{k+1}} \quad < \quad -\frac{1}{a_k}$$

$$\Rightarrow 2 - \frac{1}{a_{k+1}} \quad < \quad 2 - \frac{1}{a_k}$$

Hence, P(k+1) is true.

$$\Rightarrow a_{k+2} < a_{k+1}$$

So by Principle of Mathematical Induction , P(n) is true for all n.
$\Rightarrow a_{n+1} < a_n$ for all n
$\Rightarrow (a_n)$ is decreasing.

we will show that, $a_n > 1$ for all n. $n = 1, a_n = \frac{3}{2} > 1$
Let P(k): $a_k > 1$ for all k.

$$\Rightarrow -a_k < -1$$
$$\Rightarrow -\frac{1}{a_k} > -\frac{1}{1}$$
$$\Rightarrow 2 - \frac{1}{a_k} > 2 - \frac{1}{1}$$
$$\Rightarrow a_{k+1} > 1$$

Hence, by Principle of Mathematical Induction $a_n > 1$ for all n.

So, (a_n) is decreasing and bounded below sequence.

Hence, by Monotonic Convergence Theorem (a_n) is convergent.

18. Let (x_n) be a Cauchy

Sequence.Let $\varepsilon > 1$

Since, (x_n) is Cauchy,

so, for $\varepsilon = 1$, there exists a $k \in \mathbf{N}$ such that $|x_n - x_m| < \varepsilon = 1$ for all $n, m \geq k$

In particular for $m = k$,we have

$|x_n - x_k| < 1$ for all $n \geq k$

$\Rightarrow |x_n| < 1 + |x_k|$ for all $n \geq k$

Let, $M = Max\{|x_1|, |x_2|, |x_3|, \ldots |x_k|, 1 + |x_k|\}$

$\Rightarrow |x_n| \geq M$ for all $n \in \mathbf{N}$

$\Rightarrow (x_n)$ is bounded sequence.

Hence, Every Cauchy sequence is a bounded sequence.

19. $\sum r^n = 1 + r + r^2 + r^3 + r^4 + \ldots$

$$S_1 = 1$$

$$S_2 = 1 + r$$

$$S_3 = 1 + r + r^2$$

$$S_4 = 1 + r + r^2 + r^3$$

$$\cdot$$

$$S_n = 1 + r + r^2 + \ldots + r^{n-1}$$

$$S_n(1 - r) = (1 + r + r^2 + \ldots + r^{n-1})(1 - r)$$

$$= 1 + r + r^2 + \ldots + r^{n-1} - (r + r^2 + r^3 + \ldots + r^n)$$

$$= 1 - r^n$$

$$S_n(1 - r) = 1 - r^n$$

$$\Rightarrow S_n = \frac{1 - r^n}{1 - r}$$

$$\Rightarrow S_n = \frac{1}{1 - r} - \frac{r^n}{1 - r}$$

$$\Rightarrow S_n - \frac{1}{1 - r} \leq \frac{|r|^n}{1 - r}$$

$\Rightarrow S_n \to \frac{1}{1-r}$ as $n \to \infty$ if $|r| < 1$

$\Rightarrow \sum r^n$ is convergent and $\sum r^n = \frac{1}{1-r}$, $|r| < 1$.

20. $x_n = e^{-n}$

$\sum e^{-n^2}$
We know that , $e^x > x$ for all $x > 0$
$\Rightarrow e^{n^2} > n^2$

$\Rightarrow \frac{1}{e^{n^2}} < \frac{1}{n^2}$

We know that, $\Sigma \frac{1}{n^p}$ is a convergent series for $p \geq 1$

So $\Sigma \frac{1}{n^2}$ is a p-series.

From comparison test

$\Rightarrow \sum \frac{1}{e^{n^2}}$ is also convergent.

$\Rightarrow \sum e^{-n^2}$ is convergent series.