Theory of Real Functions

Question:1- (x_n) is any sequence that converges to a number c, then (x_n^2) converges to _ _ _ _ _ _

(a) c (b) c^2 (c) x^2 (d) x

Answer-(b)

Question:2- Suppose (x_n) is a sequence in A with

lim (x_n) = c and $x_n \neq$ c for all n .The sequence (f (x_n)) converges to

(a) c (b) A (c) L (d) None of the above

Answer-(c)

QUESTION 3 ;What is the first condition to prove sequential criterion theorem?

- 1. $\lim_{x \to 0} x = 0$
- 2. $\lim_{n \to \infty} a_n = c$
- 3. $\lim_{n \to \infty} f(a_n) = L$

Answer-(b)

QUESTION 4;The Sequential Criterion for a Limit of a Function says that as *n* goes to zero, the function *f* evaluated at these a_n will have its limit go to *L .Whether the above statement is*

1.true 2.false

Answer-(2)

QUESTION 5;In divergence criterion for limit,

1. $\lim_{n \to \infty} f(a_n) = L$

2. $\lim_{n \to \infty} f(a_n) \neq L$

Answer-(2)

QUESTION 6;Convergence test only *aplicaple* on

1.finite series

2.infinite series

Answer-(2)

Question:7- Which of the following is true for Divergence Criteria

(i)that a certain number is not the limit of a function at a point,

(ii) that the function does not have a limit at a point.

(a) only (i) (b) only (ii)
(c) both (i) & (ii) (d) None

Answer-(c)

Question:8- Which of the following has divergent limit

(a) $lim_{x\to 0}$ (1/x) (b) $lim_{x\to 0}$ (x^2/ |x|)

(c) $lim_{x\to 0}$ [Sin(x)/x] (d) All of the above

Answer-(a)

Question:9- f: A→**R** and let c be a cluster point of A. Then

(i) $lim_{x\to c} f = L$

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq$ c for all n ∈ N, the sequence (f(x_n)) converges to L the above statements are

(a)Equivalent (b) unequal

(c) equal (d) None of these

Answer-(a)

Question:10-Which of the following equation is true?

(a)f(x) ≤g(x) ≤h(x) (b)f(x) <g(x) ≤h(x)

(c)f(x) ≤g(x) <h(x) (d)f(x) <g(x) <h(x)

Answer-(a)

Question:11-Let A⊆R , let f:A->R,and
let$R$, *where c is the cluster point of both of the sets* $A \cap (c, \infty)$

(a)Left –hand limit of f at "c" (b)infinite limit (c) Right –hand limit of f at "c"
(d) None of the above

Answer-(c)

QUESTION 12;Given; $lim_{x\to c} f = +\infty$, then $lim_{x\to c} g = +\infty$.

Also,A ⊆ **R**, where f , g: A→**R**, and c € **R** is a cluster point of A (x≠c). What is the relation between f(x) and g(x)?

a. f(x) = g(x) b. f(x) < g(x)

c. f(x) ≤ g(x) d. f(x) ≥ g(x)

Answer-(c)

QUESTION 13;Given g(x)=1/x for x≠0. We know that $lim_{x\to 0} g$ does not exist.

Then $lim_{x \to 0^-}$ (1/x) = ?

a. ∞ b. -∞

c. 0 d. 1

Answer-(b)

QUESTION 14) If $lim_{x \to a} f(x)=l$ and $l \neq 0$,

then $lim_{x \to a} 1/f(x)$ = ?

a. l^2 b. l

c. 1/l d. 0

Answer-(c)

QUESTION 15) If $lim_{x \to \infty} f_1(x)= \infty$ and

$lim_{x \to \infty} f_2(x)= \infty$,

then $lim_{x \to \infty} [f_1(x)+f_2(x)]$=?

a. 0 b. ∞

c. 1 **d. -1**

Answer-(b)

ANALYSIS(2)

QUESTION 1: Determine the limit

$$\lim_{x\to 2}(x^3 - 5),\ x \in R.$$

Solution: We have

[using the theorem of limits]

$$\lim_{x\to 2}(x^3 - 5) = \lim_{x\to 2}(x^3) - \lim_{x\to 2}(5)$$

$$= (2^3) - 5$$

$$= 8 - 5 = 3$$

Hence, $\lim_{x\to 2}(x^3 - 5) = 3,\ x \in R$

QUESTION 2: Determine the limit

$$\lim_{x\to 1}(x + 2)(3x - 1),\ x \in R.$$

Solution: We have

$$\lim_{x\to 1}(x+2)(3x-2) = \lim_{x\to 1}(x+2).\lim_{x\to 1}(3x-2)$$ [using the theorem of limits]

$$= (1+2)(3-2)$$

$$= 3$$

Hence, $\lim_{x\to 1}(x + 2)(3x - 1) = 3,\ x \in R$

QUESTION 3: Determine the limit

SOLUTION;WE HAVE

$$\lim_{x\to 2}\sqrt{\frac{4x+1}{x+2}}.$$

$$\lim_{x\to 2}\sqrt{\frac{4x+1}{x+2}}=\sqrt{\lim_{x\to 2}\left(\frac{4x+1}{x+2}\right)}$$ [using the theorem of limits]

$$=\sqrt{\frac{\lim_{x\to 2}(4x+1)}{\lim_{x\to 2}(x+2)}}$$ [using the theorem of limits]

$$=\sqrt{\frac{\lim_{x\to 2}(4x)+\lim_{x\to 2}(1)}{\lim_{x\to 2}(x)+\lim_{x\to 2}(2)}}$$

$$=\sqrt{\frac{(4\text{x}2+1)}{(2+2)}}$$

$$=\sqrt{\frac{9}{4}}=\frac{3}{2}$$

Hence, $\lim_{x\to 2}\sqrt{\frac{4x+1}{x+2}}=\frac{3}{2}$.

QUESTION 4 PROOF If f(x) is a polynomial function, then $\lim_{x\to 0} f(x) = f(x_0)$.

Proof: Let f(x) be a polynomial function of degree n in x on R such that

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n \qquad (a_0 \neq 0) \qquad (1)$$

thus $f(x_0) = a_0x_0^n + a_1x_0^{n-1} + a_2x_0^{n-2} + \dots + a_{n-1}x_0 + a_n \qquad (a_0 \neq 0)$ (2)

We have

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0}\left(a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n\right), \quad (a_0 \neq 0)$$

$$= \lim_{x\to x_0}\left(a_0x^n\right) + \lim_{x\to x_0}\left(a_1x^{n-1}\right) + \lim_{x\to x_0}\left(a_2x^{n-2}\right) + \dots + \lim_{x\to x_0}\left(a_{n-1}x\right) + \lim_{x\to x_0}\left(a_n\right)$$

$$= a_0\lim_{x\to x_0}\left(x^n\right) + a_1\lim_{x\to x_0}\left(x^{n-1}\right) + a_2\lim_{x\to x_0}\left(x^{n-2}\right) + \dots + a_{n-1}\lim_{x\to x_0}\left(x\right) + \lim_{x\to x_0}\left(a_n\right)$$

$$\Rightarrow \quad \lim_{x\to x_0} f(x) = a_0x_0^n + a_1x_0^{n-1} + a_2x_0^{n-2} + \dots + a_{n-1}x_0 + a_n \qquad (3)$$

Using equation (2) and (3) we have

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Hence for a polynomial f(x) of degree n

$$\lim_{x \to x_0} f(x) = f(x_0) \text{ .}$$

QUESTION 5 ;PROOF IF X be a non-empty subset of R and let $f : X \to R$. If $a \le f(x) \le b \; for \; all \; x \in X, x \ne x_0$ and if $\lim_{x \to x_0} f(x)$ exists. Then $a \le \lim_{x \to x_0} f(x) \le b$.

Proof: Let $\lim_{x \to x_0} f(x)$ exists and let

$$\lim_{x \to x_0} f(x) = \ell \qquad (1)$$

Then by the sequential criteria it follows that if $< x_n >$ is any sequence of real numbers such that $x_0 \ne x_n \in X \; for \; all \; n \in N$ and if the sequence $< x_n >$ converges to x_0, then the sequence $f(x_n)$ converges to ℓ .

Since

$$a \le f(x_n) \le b \; for \; all \; n \in N$$

Then using the theorem that if $< x_n >$ is a convergent sequence and if $a \le x_n \le b \; for \; all \; n \in N, \; then \; a \le \lim(x_n) \le b$.

$$\Rightarrow \qquad a \le \lim f(x_n) \le b$$

or we can say that

$$a \le \lim_{x \to x_0} f(x) \le b \text{ .}$$

Hence Proved.

QUESTION 6: Show that $\lim_{x \to x_0} x^{3/2} = 0, \quad (x > 0)$ **.**

Solution: Let $f(x) = x^{3/2} \; for \; x > 0$

Now we know that the inequality

$$x \le x^{1/2} \le 1 \; holds \; for \; 0 < x \le 1$$

$\Rightarrow \quad x^2 \le x^{3/2} \le x \; for \; 0 < x \le 1$ since lim x-0 x^2=0. Thus applying the squeeze theorem on equation (1), we have lim x-0 x^3/2=0

Question 7: Show that $\lim_{x \to 0} \sin x = 0$ **.**

Solution: We know that

$$-1 \le \cos x \le 1 \qquad for\ all\ x \ge 0$$

on integrating both sides w.r.t. x, between the limits 0 and x.

$$\Rightarrow \quad -\int_0^x dx \le \int_0^x \cos dx \le \int_0^x dx$$

$$\Rightarrow \quad -x \le \sin x \le x \quad for\ all\ x \ge 0$$

since

$$\lim_{x\to 0}(-x) = -\lim_{x\to 0}(x) = 0$$

and $\lim_{x\to 0}(x) = 0$

thus applying the squeeze theorem we have

$$\lim_{x\to 0}(\sin x) = 0 \text{ .}$$

. —— $_{x\to 0}$ | x |

QUESTIONS;8 lim X_-0 (SIN X/X)=1
Solution: We know that

$$-1 \le \cos x \le 1 \qquad for\ all\ x \ge 0 \qquad (1)$$

on integrating both sides w.r.t. x, between the limits 0 and x.

$$\Rightarrow \quad -\int_0^x dx \le \int_0^x \cos dx \le \int_0^x dx$$

$$\Rightarrow \quad -x \le \sin x \le x \quad for\ all\ x \ge 0$$

on again integrating both sides w.r.t. x, between the limits 0 and x.

$$-\int_0^x x dx \le \int_0^x \sin dx \le \int_0^x x dx$$

$$\Rightarrow \quad -\frac{x^2}{2} \le -\cos x + 1 \quad + \le x/2^2$$

$\cos x \;\; 1$

$$\Rightarrow \quad 1-\frac{x^2}{2} \le \cos x \le 1+\frac{x^2}{2} \qquad (2)$$

using equation (1) and (2), we have

$$1-\frac{x^2}{2} \le \cos x \le 1 \qquad (3)$$

On integrating equation (3) w.r.t. x, between the limits o and x, we have

$$x-\frac{x^3}{6} \le \sin x \le x \quad \textit{for all } x \ge 0 \qquad (4)$$

and $$x \le \sin x \le x-\frac{x^3}{6} \quad \textit{for all } x < 0 \qquad (5)$$

Thus from (4) and (5) we conclude that

$$1-\frac{x^2}{6} \le \frac{\sin x}{x} \le 1 \quad \textit{for all } x \ne 0$$

Since

$$\lim_{x\to 0}\left(1-\frac{x^2}{6}\right) = 1-\frac{1}{6}\lim_{x\to 0}(x^2) = 1-0 = 1$$

and $\lim_{x\to 0}(1) = 1$

thus applying the squeeze theorem we conclude that

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1.$$

Question 9: Check the continuity of the following function at the origin:

$$F(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution: Here $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \dfrac{xe^{1/x}}{1+e^{1/x}} = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{e^{-1/x}+1} = 0$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 0$$

So the function is continuous at the origin:

Question 10 Define the function defined below for continuity at

x = 0 where

$f(x) = \dfrac{\sin^2 ax}{x^2}$ for $x \neq 0$, and $f(x) = 1$ for x = 0.

Solution: We have f (0) = 1

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{\sin^2 ah}{h^2}$$

$$= \lim_{h \to 0} \left(\frac{\sin ah}{ah} \right)^2 . a^2 = 1.a^2 = a^2$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{\sin^2(-ah)}{(-h)^2}$$

$$= \lim_{h \to 0} \frac{\sin^2 ah}{h^2} = a^2$$

We know f (x) is continuous at x = 0 iff

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x)$$

Hence $f(x)$ is discontinuous at $x = 0$ unless $a = 1$.

Question 11: Check the continuity of the function

$$f(x) = \begin{cases} \dfrac{|x-a|}{x-a}, & x \neq a \\ 1 & ; x = a \end{cases}$$ at $x = a$.

Solution:- Let

$$f(x) = \begin{cases} \dfrac{(x-a)}{x-a}, & \text{if } x > a \\ -\dfrac{(x-a)}{x-a}, & \text{if } x < a \\ 1, & \text{if } x = a \end{cases}$$

$$\lim_{x\to a^-} f(x) = \lim_{x\to a} \frac{-(x-a)}{x-a} = -1$$ as $x \neq a$

$$\lim_{x \to a^+} f(x) = \lim_{x \to a} \frac{(x-a)}{x-a} = 1 \text{ as } x \neq a$$

Thus $\lim_{x \to a} f(x)$ does not exist and so *f* is discontinuous at $x = a$.

QUESTION 12: Show that f : R $\to$ R given $f(x) = x^2$ is continuous but not by uniformly continuous on R.

Solution: We have $f(x) = x^2, \ \forall x \in \mathrm{R}$

Let $a \in$ R, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} x^2 = a^2 = f(a)$$

Therefore, $f(x)$ is continuous at „*a*".

But „*a*' was an arbitrary point of R. Therefore $f(x)$ is continuous at each point of R and hence $f(x)$ is continuous on R.

$f(x) = x^2$ is not uniformly continuous on R is already shown in example 7.

Value Addition: Remark:
If a function *f* is uniformly continuous on an interval I, then it is continuous on I i.e., it is continuous at each point on I. Thus Uniform continuity always implies continuity. But the converse is not true. A function may be continuous in an open interval I but it may fail to be uniformly continuous in I. For example, the function $f(x) = \frac{1}{x}, \forall x \in]0,1[$ is continuous in] 0, 1 [but it is not uniformly continuous in] 0, 1 [. But if a function *f* is continuous in a closed and bounded interval [*a*, *b*], then it is uniformly continuous in [*a*, *b*].

QUESTION 13: PROVE THAT If a function *f* is uniformly continuous on an Interval I, then it is continuous on I.

Proof:- Let x_0 be any point of I and let $\varepsilon > 0$ be given.

Since f is uniformly continuous on I, therefore, for $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta, \forall \; x, y \in \mathrm{I}$.

Let $y = x_0$

$\Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta \; \forall \; x \in \mathrm{I}$.

$\Rightarrow f$ is continuous at x_0.

But x_0 is an arbitrary point of I, therefore f is continuous at every point of I. Hence f is continuous on I.

NOTE:- The converse of the above theorem is not true. For example, Consider the function $f(x) = x^2, \forall x \in R$ which is continuous for all $x \in$ R but not uniformly continuous. The $f(x) = \frac{1}{x}$ is uniformly continuous on converse can only hold if the function is continuous on a closed and bounded interval.

QUESTION 14: Show that the function

the set A = [a, ∞) where „a' is a positive constant.

Solution: Given $f(x) = \frac{1}{x}$

Let ε > 0 be given

$$\left|f(x_1) - f(x_2)\right| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right| = \frac{|x_2 - x_1|}{|x_1||x_2|}$$

As $x_1, x_2 \in [a,\infty) \Rightarrow a < x_1,\ a < x_2$

$$\Rightarrow \quad \frac{1}{x_1}, \frac{1}{x_2} < \frac{1}{a}$$

$$|f(x_1) - f(x_2)| = \frac{|x_1 - x_2|}{|x_1||x_2|} < \frac{1}{a^2}|x_1 - x_2| < \varepsilon$$

if $\quad |x_1 - x_2| < a^2\varepsilon.$

Take $\delta = a^2\varepsilon$

$$\Rightarrow \quad |f(x_1) - f(x_2)| < \varepsilon \text{ if } |x_1 - x_2| < \delta$$

Therefore „*f* “ is uniformly continuous on A.

QUESTION 15: Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on A = [1, ∞), but that it is not uniformly continuous on B = (0, ∞).

Solution: Let $f(x) = \frac{1}{x^2}$ on A = [1, ∞)

Let $x_1, x_2 \in A$ and $\varepsilon > 0$ be given

$$|f(x_1) - f(x_2)| = \left|\frac{1}{x_1^2} - \frac{1}{x_2^2}\right| = \frac{1}{x_1^2 x_2^2}|x_1 - x_2||x_1 + x_2|$$

$$\leq \frac{1}{x_1 x_2}|x_1 - x_2||x_1 + x_2| \qquad \left[\because x_{1,2} \geq 1 \text{ and } \frac{1}{x_1}, \frac{1}{x_2} \leq 1\right]$$

$$= |x_1 - x_2|\left|\frac{1}{x_1} + \frac{1}{x_2}\right|$$

$$\leq |x_1 - x_2|\left(\left|\frac{1}{x_1}\right| + \left|\frac{1}{x_2}\right|\right)$$

$$\leq |x_1 - x_2| (1+1)$$

$$= 2|x_1 - x_2|$$

$$< \varepsilon \text{ if } |x_1 - x_2| < \frac{\varepsilon}{2}$$

Let us choose δ = ε/2

Then $|f(x_1) - f(x_2)| < \varepsilon$ if $|x_1 - x_2| < \delta$.

Therefore *f* is uniformly continuous on A.

Now given B = (0, ∞)

Take $x_1 = \frac{1}{\sqrt{n}}$ and $x_2 = \frac{1}{\sqrt{n+1}}$

$$|f(x_1) - f(x_2)| = \left|\frac{1}{x_1^2} - \frac{1}{x_2^2}\right| = |n - \overline{n+1}| = 1$$

Taking $\varepsilon = \frac{1}{2}$ and δ such that $\delta > \frac{1}{2n}$

Now $|f(x_1) - f(x_2)| > \varepsilon$

$$|x_1 - x_2| = \left|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right| = \left|\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}\sqrt{n+1}}\right| \times \frac{\left(\sqrt{n+1} + \sqrt{n}\right)}{\left(\sqrt{n+1} + \sqrt{n}\right)}$$

$$= \frac{n+1-n}{\sqrt{n}\sqrt{n+1}\left[\sqrt{n+1} + \sqrt{n}\right]}$$

$$< \frac{1}{2\sqrt{n}\sqrt{n}} = \frac{1}{2n} < \delta$$

Therefore $|x_1 - x_2| < \delta$ but $|f(x_1) - f(x_2)| > \varepsilon$

$\Rightarrow$ f is not uniformly continuous on B.

QUESTION 16: Show that the $f(x) = \dfrac{1}{1+x^2}$ for $x \in$ R is uniformly function continuous on R.

Solution: Given $f(x) = \dfrac{1}{1+x^2}$ where $x_1, x_2 \in \mathbb{R}$; let ε > 0 be given.

$$|f(x_1) - f(x_2)| = \left| \frac{1}{1+x_1^2} - \frac{1}{1+x_2^2} \right|$$

$$= \left| \frac{x_2^2 - x_1^2}{(1+x_1^2)(1+x_2^2)} \right|$$

$$= \frac{|x_2^2 - x_1^2|}{|1+x_1^2||1+x_2^2|} = \frac{|x_2 - x_1||x_2 + x_1|}{(1+x_1^2)(1+x_2^2)}$$

$$\leq \frac{|x_2 - x_1|(|x_2| + |x_1|)}{(1+x_1^2)(1+x_2^2)}$$

$$= |x_2 - x_1| \left[\frac{|x_2|}{(1+x_1^2)(1+x_2^2)} + \frac{|x_1|}{(1+x_1^2)(1+x_2^2)} \right]$$

$$\left\{ \text{Now for } \frac{|x_1|}{(1+x_1^2)(1+x_2^2)} \right.$$

We can say $|x_1| < 1 + x_1^2$ ----(i)

$1 < 1 + x_2^2$ -----(ii)

Adding (i) and (ii)

We get $|x_1| < [(1+x_1^2)(1+x_2^2)]$

$$\Rightarrow \quad \frac{|x_1|}{(1+x_1^2)(1+x_2^2)} < 1$$

Similarly $\frac{|x_2|}{(1+x_1^2)(1+x_2^2)} < 1$

$$\Rightarrow \quad |f(x_1) - f(x_2)| \le |x_2 - x_1|(1+1) = 2|x_2 - x_1| < \varepsilon \quad \text{if} \quad |x_2 - x_1| < \frac{\varepsilon}{2}.$$

Let us choose $\delta = \frac{\varepsilon}{2}$

Then $|f(x_1) - f(x_2)| < \varepsilon$ if $|x_2 - x_1| < \delta$

Therefore *f* is uniformly continuous on R.

QUESTION 17: Show that if *f* and *g* are uniformly continuous on a subset A of R, then *f* + *g* is uniformly continuous on A.

Solution: Let *f* and *g* be uniformly continuous on A

Let $\varepsilon > 0$ be given and let $x_1, x_2 \in$ A.

Since *f* and *g* are uniformly continuous

Therefore, $|f(x_2) - f(x_1)| < \frac{\varepsilon}{2}$ when $|x_2 - x_1| < \delta_1$ .... (1)

and $|g(x_2) - g(x_1)| < \frac{\varepsilon}{2}$ when $|x_2 - x_1| < \delta_2$ .... (2)

Let $\delta = \min\{\delta_1, \delta_2\}$

Therefore (1) and (2) hold for δ.

$$\Rightarrow \quad |(f+g)x_2 - (f+g)x_1| \le |f(x_2) - f(x_1)| + |g(x_2) - g(x_1)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore, $|(f+g)x_2 - (f+g)\,x| < \varepsilon$ when $|x_1 - x_2| < \delta \quad \forall\, x_1, x_2 \in A$.

Hence $f + g$ is uniformly continuous in A.

QUESTION 18: Show that if f and g are uniformly continuous on $A \subseteq R$ and if they are both bounded on A, then their product fg is uniformly continuous on A.

Solution: For given $\varepsilon > 0$ there exists $\delta_1, \delta_2 > 0$ such that

$$|f(x_2) - f(x_1)| < \frac{\varepsilon}{2k} \quad \text{whenever} \quad |x_2 - x_1| < \delta_1 \qquad .... \quad (1)$$

and $$|g(x_2) - g(x_1)| < \frac{\varepsilon}{2k} \quad \text{whenever} \quad |x_2 - x_1| < \delta_2 \qquad .... \quad (2)$$

Given f and g are bounded there exists $k_1 > 0$ and $k_2 > 0$ such that

$$|f(x)| \le k_1 \quad \text{and} \quad |g(x)| \le k_2 \qquad \forall\, x \in A$$

Let k = max . (k_1, k_2)

Therefore $|f(x)| \le k$ and $|g(x)| \le k$.

$$|(fg)(x_2) - (fg)(x_1)| = |f(x_2)g(x_2) - f(x_1)g(x_1)|$$

$$= |(f(x_2) - f(x_1))g(x_2) + f(x_1)(g(x_2) - g(x_1))|$$

$$\le |g(x_2)||f(x_2) - f(x_1)| + |f(x_1)||g(x_2) - g(x_1)|$$

Let δ = min . (δ_1, δ_2) .... (3)

Therefore, $|(fg)(x_2) - (fg)(x_1)| < k.\frac{\varepsilon}{2k} + k.\frac{\varepsilon}{2k} = \varepsilon$ whenever $|x_2 - x_1| < \delta$

(From (1), (2) and (3))

$\Rightarrow$ fg is uniformly continuous on A.

Value Addition: Note
The product of two uniformly continuous functions may not be uniformly continuous.

QUESTION 19: If $f(x) = x$ and $g(x) = \sin x$, show that both *f* and *g* are uniformly continuous on R, but that their product *fg* is not uniformly continuous on R.

Solution: Given $f(x) = x$ and $g(x) = \sin x$

$$|f(x_2) - f(x_1)| = |x_2 - x_1| < \delta$$

Choose $\delta = \varepsilon$, so we get

$|f(x_2) - f(x_1)| < \varepsilon$ whenever $|x_2 - x_1| < \delta$.

Therefore *f* is uniformly continuous on R.

$$|g(x_2) - g(x_1)| = |\sin x_2 - \sin x_1|$$

$$= 2\left|\sin\frac{x_2 - x_1}{2}\cos\frac{x_2 + x_1}{2}\right|$$

$$\le 2 \times \frac{x_2 - x_1}{2} \times 1 \qquad [\text{Because } \cos\theta \le 1 \text{ and } \sin\theta \le \theta]$$

$$\le (x_2 - x_1)$$

$$< \delta$$

If we choose $\delta = \varepsilon$, we get

$|g(x_2) - g(x_1)| < \varepsilon$ whenever $|x_2 - x_1| < \delta$

Therefore *g* is uniformly continuous on R.

Since *f* is not bounded on R, therefore by the previous example we can easily prove that the product *fg* is not uniformly continuous on A.

QUESTION 20: Prove that if *f* and *g* are each uniformly continuous on R, then the composite function *f o g* is uniformly continuous on R.

Solution: Let $x_1, x_2 \in \mathrm{R}$ and $\varepsilon > 0$ be given

$$\left|(fog)(x_2) - (fog)(x_1)\right| = \left|f\left[g(x_2)\right] - f\left[g(x_1)\right]\right|.$$

Here $g(x_1)$ and $g(x_2) \in$ R and *f* is uniformly continuous on R, therefore there exist $\delta_1 > 0$ such that

$$\left|f\left[g(x_2)\right] - f\left[g(x_1)\right]\right| < \varepsilon \quad \text{whenever} \quad |\, g(x_2) - g(x_1) \,| < \delta_1$$

Also *g* is uniformly continuous on R and $x_1, x_2 \in \mathrm{R}$

Therefore there exist $\delta_2 > 0$ such that

$$\left|g(x_2) - g(x_1)\right| < \delta_1 \quad \text{whenever} \quad |x_2 - x_1 < \delta_2$$

Therefore, $\left|(fog)(x_2) - (fog)(x_1)\right| < \varepsilon$ whenever $|x_2 - x_1| < \delta_2$

$\Rightarrow$ *f o g* is uniformly continuous on R.

$f(x) = \sin x^2$ for uniform continuity

QUESTION 21 CHECKk the

function on [0, ∞ [.

Solution: Let ε = 1/2 > 0 and δ > 0 be there.

We can choose a positive integer „*n*' such that

$$n > \pi \mid \delta^2 \qquad \ldots. \quad (1)$$

Let $x_1 = \sqrt{\dfrac{n\pi}{2}}$; $x_2 = \sqrt{(n+1)\dfrac{\pi}{2}} \in [0, \infty[$

$$\left|f(x_2) - f(x_1)\right| = \left|\sin x_2^2 - \sin x_1^2\right|$$

$$= \left|\sin(n+1)\frac{\pi}{2} - \sin\frac{n\pi}{2}\right|$$

$$= \begin{cases} \left|0 - (\pm 1)\right| = 1; \text{ if } n \text{ is odd.} \\ \left|\pm 1 - 0\right| = 1; \text{ if } n \text{ is even.} \end{cases}$$

so $\quad \left|f(x_2) - f(x_1)\right| = 1 > \varepsilon$

For $\left|x_2 - x_1\right| = \left|\frac{x_2^2 - x_1^2}{x_2 + x_1}\right| = \frac{\pi/2}{\sqrt{(n+1)\frac{\pi}{2}} + \sqrt{\frac{n\pi}{2}}}$

$$< \frac{\pi}{2\left\{2\sqrt{\frac{n\pi}{2}}\right\}} < \frac{\pi}{\sqrt{n\pi}} = \sqrt{\frac{\pi}{n}} < \delta \qquad \text{[By (1)]}$$

$\Rightarrow \quad \left|f(x_2) - f(x_1)\right| > \varepsilon$, when $\left|x_2 - x_1\right| < \delta$

Hence $f(x) = \sin x^2$ is not uniformly continuous on [0, ∞ [.

QUESTION 22: Show that $f(x) = \sqrt{x}$ is uniformly continuous in [0, 1].

Solution: Let $x, y \in [0,1]$ where $x > y \geq 0$

$$\left|f(x) - f(y)\right| = \left|\sqrt{x} - \sqrt{y}\right| = \left|\frac{\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)}{\sqrt{x} + \sqrt{y}}\right|$$

$$= \frac{x - y}{\sqrt{x} + \sqrt{y}} \leq \frac{x - y}{\sqrt{x}} \leq \frac{x - y}{\sqrt{x - y}} = \sqrt{x - y}$$

$\Rightarrow \quad \left|f(x) - f(y)\right| \leq \sqrt{x - y}$.

Let ε > 0 be given

So $\left|f(x) - f(y)\right| < \varepsilon$; when $\sqrt{x - y} < \varepsilon$

$\Rightarrow \quad \left|f(x) - f(y)\right| < \varepsilon$; when $\left|x - y\right| < \delta\left(= \varepsilon^2\right) \; \forall \; x, y \in [0,1]$

$\Rightarrow$ f is uniformly continuous in $[0,1]$.