Multivariate Calculus

Question.1

Let $f(x,y) = x^2\sqrt{y} + 2x + 1$. Find $f(0,0), f(-1,1), f(t^2,t), f(2a,b)$. Also find the domain of f.

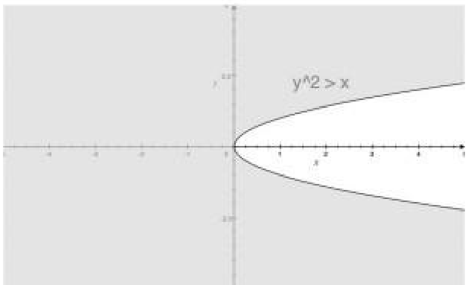
Solution: By substitution

$$\begin{aligned} f(0,0) &= 0^2\sqrt{0} + 2\cdot 0 + 1 = 1 \\ f(-1,1) &= (-1)^2\sqrt{1} + 2(-1) + 1 = 0 \\ f(t^2,t) &= (t^2)^2\sqrt{t} + 2t^2 + 1 = t^{9/2} + 2t^2 + 1 \\ f(2a,b) &= (2a)^2\sqrt{b} + 2(2a) + 1 = 4a^2\sqrt{b} + 4a + 1 \end{aligned}$$

Question.2

Sketch the domain of the function $f(x,y) = \log(y^2 - x)$.

Solution: $\log(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, for $y^2 > x$. We first sketch the parabola $y^2 = x$. The region $y^2 > x$ consists of all the points above and below this curve.



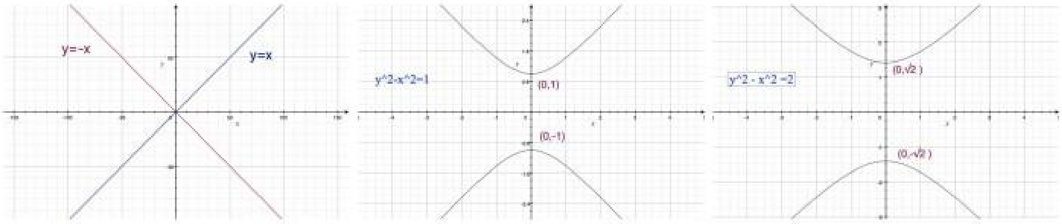
Question.3

Draw the level curves of $f(x, y) = y^2 - x^2$ of height $k = 0, 1, 2, \ldots$.

Solution: The level curves of the surface $f(x, y) = y^2 - x^2$ of height $k = 0, 1, 2$ are respectively

$$y^2 - x^2 = 0, \quad y^2 - x^2 = 1, \quad y^2 - x^2 = 2$$

While $y^2 - x^2 = 0$ represents a pair of straight lines $y = x$ and $y = -x$, $y^2 - x^2 = 1$ and $y^2 - x^2 = 2$ represent hyperbolas.



Question.4

Use the $\epsilon - \delta$ definition of limit to prove $\lim_{(x,y)\to(0,0)} (x^2 + y^2)^{\frac{3}{2}} = 0$.

Solution:

Let $\epsilon > 0$ be given.

(To show there exists a number $\delta > 0$ such that $0 < \sqrt{x^2 + y^2} < \delta \Rightarrow |f(x, y) - L| < \epsilon$)

$$\begin{aligned} |f(x, y) - L| &= |(x^2 + y^2)^{\frac{3}{2}} - 0| \\ &= (x^2 + y^2)^{\frac{3}{2}} < \epsilon \\ &\text{if } (x^2 + y^2)^{\frac{1}{2}} < \epsilon^{\frac{1}{3}} \end{aligned}$$

Taking $\delta = \epsilon^{\frac{1}{3}}$, we have

$$0 < \sqrt{x^2 + y^2} < \delta \Rightarrow |(x^2 + y^2)^{\frac{3}{2}} - 0| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we have $\lim_{(x,y)\to(0,0)} (x^2 + y^2)^{\frac{3}{2}} = 0$.

Question.5

Use the $\epsilon - \delta$ definition of limit to prove $\lim_{(x,y)\to(0,0)} y\ Sin\frac{1}{x} = 0.$

Solution:

Let $\epsilon > 0$ be given.

(To show there exists a number $\delta > 0$ such that $0 < \sqrt{x^2 + y^2} < \delta \Rightarrow |f(x, y) - L| < \epsilon$)

$$\begin{aligned} |f(x, y) - L| &= |y\ Sin\frac{1}{x} - 0| \\ &\leq |y| \\ &\leq \sqrt{x^2 + y^2} \end{aligned}$$

Taking $\delta = \epsilon$, we have
$0 < \sqrt{x^2 + y^2} < \delta \Rightarrow |y\ Sin\frac{1}{x} - 0| < \epsilon.$
Since $\epsilon > 0$ was arbitrary, we have $\lim_{(x,y)\to(0,0)} y\ Sin\frac{1}{x} = 0.$

Question.6

Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + x - xy - y}{x - y}.$

Solution: First, note that for $x \neq y$

$$f(x, y) = \frac{x^2 + x - xy - y}{x - y} = \frac{(x + 1)(x - y)}{x - y} = x + 1$$

Therefore, since $f(x, y)$ is defined only for $x \neq y$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + x - xy - y}{x - y} = \lim_{(x,y)\to(0,0)} (x + 1) = 1$$

Question.7

$$\text{If } f(x,y) = x^3y + x^2y^2, \text{ find:}$$

a. f_x b. f_y

Solution: a. For f_x, hold y constant and find the derivative with respect to x :

$$f_x(x,y) = 3x^2y + 2xy^2$$

b. For f_y, hold x constant and find the derivative with respect to y :

$$f_y(x,y) = x^3 + 2x^2y$$

Question.8

$$\text{Let } z = x^2 \sin\left(3x + y^3\right)$$

a. Evaluate $\left.\frac{\partial z}{\partial x}\right|_{(\pi/3,0)}$

b. Evaluate z_y at (1,1).

Solution:

a. $\frac{\partial z}{\partial x} = 2x\sin\left(3x+y^3\right) + x^2\cos\left(3x+y^3\right)(3)$

$$= 2x\sin\left(3x+y^3\right) + 3x^2\cos\left(3x+y^3\right)$$

Thus,

$$\left.\frac{\partial z}{\partial x}\right|_{(\pi/3,0)} = 2\left(\frac{\pi}{3}\right)\sin\pi + 3\left(\frac{\pi}{3}\right)^2\cos\pi = \frac{2\pi}{3}(0) + \frac{\pi^2}{3}(-1) = -\frac{\pi^2}{3}$$

b. $z_y = x^2\cos\left(3x+y^3\right)\left(3y^2\right) = 3x^2y^2\cos\left(3x+y^3\right)$ so that

$$z_y(1,1) = 3(1)^2(1)^2\cos(3+1) = 3\cos 4$$

Question.9

Verify that $T(x,t) = e^{-t}\cos\frac{x}{c}$ satisfies the **heat equation**, $\frac{\partial T}{\partial t} = c^2\frac{\partial^2 T}{\partial x^2}$.

Solution:

$$\frac{\partial T}{\partial t} = -e^{-t}\cos\frac{x}{c}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{1}{c}e^{-t}\sin\frac{x}{c}\right)$$

$$= -\frac{1}{c^2}e^{-t}\cos\frac{x}{c}$$

Thus, T satisfies the heat equation $\frac{\partial T}{\partial t} = c^2\frac{\partial^2 T}{\partial x^2}$.

Question.10

Determine f_{xy}, f_{yx}, f_{xx} and f_{xxy} where $f(x,y) = x^2ye^y$.

Solution: We have the partial derivatives

$$f_x = 2xye^y \quad f_y = x^2e^r + x^2ye^r$$

The mixed partial derivatives (which must be the same by the previous theorem) are

$$f_{xy} = (f_x)_y = 2xe^x + 2xye^y \quad f_{yx} = (f_y)_x = 2xe^r + 2xye^r$$

Finally, we compute the second- and higher-order partial derivatives:

$$f_{xx} = (f_x)_x = 2ye^y \quad \text{and} \quad f_{xy} = (f_{xx})_y = 2e^y + 2ye^y$$

Question.11

Using increments to estimate the change of a function

An open box has length 3ft, width 1ft, and height 2ft and is constructed from material that costs $\$2/\text{ft}^2$ for the sides and $\$3/\text{ft}^2$ for the bottom. Compute the cost of constructing the box, and then use increments to estimate the change in cost if the length and width are each increased by 3 in. and the height is decreased by 4 in.

Solution: An open (no top) box with length x, width y, and height z has surface area

$$S = \underbrace{xy}_{\text{Bottom}} + \underbrace{2xz + 2yz}_{\text{Four side faces}}$$

Because the sides cost $\$2/\text{ft}^2$ and the bottom $\$3/\text{ft}^2$, the total cost is

$$C(x, y, z) = 3xy + 2(2xz + 2yz)$$

The partial derivatives of C are

$$C_x = 3y + 4z \quad C_y = 3x + 4z \quad C_z = 4x + 4y$$

and the dimensions of the box change by.

$$\Delta x = \frac{3}{12} = 0.25\text{ft} \quad \Delta y = \frac{3}{12} = 0.25\text{ft} \quad \Delta z = \frac{-4}{12} \approx -0.33\text{ft}$$

Thus, the change in the total cost is approximated by

$$\begin{aligned}\Delta C &\approx C_x(3,1,2)\Delta x + C_y(3,1,2)\Delta y + C_z(3,1,2)\Delta z\\ &= [3(1) + 4(2)](0.25) + [3(3) + 4(2)](0.25) + [4(3) + 4(1)]\left(-\frac{4}{12}\right)\\ &\approx 1.67\end{aligned}$$

That is, the cost increases by approximately \$1.67.

Question.12

Application of the total differential

At a certain factory, the daily output is $Q = 60K^{1/2}L^{1/3}$ units, where K denotes the capital investment (in units of \$1,000) and L the size of the labor force (in workerhours). The current capital investment is \$900,000, and 1,000 worker-hours of labor are used each day. Estimate the change in output that will result if capital investment is increased by \$1,000 and labor is decreased by 2 worker-hours.

Solution: The change in output is estimated by the total differential dQ.

We have $K = 900$, $L = 1,000, dK = \Delta K = 1$, and $dL = \Delta L = -2$. The total differential of $Q(x, y)$ is

$$\begin{aligned} dQ &= \frac{\partial Q}{\partial K}dK + \frac{\partial Q}{\partial L}dL \\ &= 60\left(\frac{1}{2}\right)K^{-1/2}L^{1/3}dK + 60\left(\frac{1}{3}\right)K^{1/2}L^{-2/3}dL \\ &= 30K^{-1/2}L^{1/3}dK + 20K^{1/2}L^{-2/3}dL \end{aligned}$$

Substituting for K, L, dK, and dL

$$dQ = 30(900)^{-1/2}(1,000)^{1/3}(1) + 20(900)^{1/2}(1,000)^{-2/3}(-2) = -2$$

Thus, the output decreases by approximately 2 units when the capital investment is increased by \$1,000 and labor is decreased by 2 worker-hours.

Question.13

Establishing differentiability

Show that $f(x, y) = x^2y + xy^3$ is differentiable for all (x, y).

Solution: Compute the partial derivatives

$$\begin{aligned} f_x(x, y) &= \tfrac{\partial}{\partial x}\left(x^2y + xy^3\right) = 2xy + y^3 \\ f_y(x, y) &= \tfrac{\partial}{\partial y}\left(x^2y + xy^3\right) = x^2 + 3xy^2 \end{aligned}$$

Because f, f_x, and f_y are all polynomials in x and y, they are continuous throughout the plane. Therefore, the sufficient condition for differentiability theorem assures us that f must be differentiable for all x and y.

Question.14

Let

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

Show that the above function is not differentiable at (0,0).

Solution: If $f(x,y)$ were differentiable at the origin, it would have to be continuous there. Thus, we can show f is not differentiable by showing that it is not continuous at (0,0). Now note along path $y=mx$,

$$\begin{aligned} f(x,y) &= \frac{x(mx)}{x^2+(mx)^2} \\ &= \frac{m}{1+m^2} \\ \text{so} \quad \lim_{(x,y)\to(0,0)} f(x,y) &= \frac{m}{1+m^2} \quad \text{along y = m x} \end{aligned}$$

Question.15

Let $z=x^2+y^2$, where $x=\frac{1}{t}$ and $y=t^2$. Compute $\frac{dz}{dt}$ in two ways:

a. by nrst expressing z explicitly in terms of t

b. by using the chain rule

Solution: a. By substituting $x=\frac{1}{t}$ and $y=t^2$, we find that

$$z=x^2+y^2=\left(\frac{1}{t}\right)^2+\left(t^2\right)^2=t^{-2}+t^4 \quad \text{for} \quad t\neq 0$$

Thus, $\frac{dz}{dt}=-2t^{-3}+4t^3$

b. Because $z=x^2+y^2$ and $x=t^{-1}, y=t^2$

$$\frac{\partial z}{\partial x}=2x; \quad \frac{\partial z}{\partial y}=2y; \quad \frac{dx}{dt}=-t^{-2}; \quad \frac{dy}{dt}=2t$$

Use the chain rule for one independent parameter:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2x)\left(-t^{-2}\right)+2y(2t) \quad \text{Using Chain rule} \\ &= 2\left(t^{-1}\right)\left(-t^{-2}\right)+2\left(t^2\right)(2t) \quad \text{by substitution} \\ &= -2t^{-3}+4t^3 \end{aligned}$$

Question.16

A right circular cylinder is changing in such a way that its radius r is increasing at the rate of 3 in./min and its height h is decreasing at the rate of 5 in./min. At what rate is the volume of the cylinder changing when the radius is 10 in. and the height is 8 in.?

Solution : The volume of the cylinder is $V = \pi r^2 h$.
We have $\frac{dr}{dt} = 3$ and $\frac{dh}{dt} = -5$. We find that

$$\frac{\partial V}{\partial r} = \pi(2r)h \quad \text{and} \quad \frac{\partial V}{\partial h} = \pi r^2(1)$$

By the chain rule for one parameter,

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi r h\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$$

Thus, at the instant when $r = 10$ and $h = 8$, we have

$$\frac{dV}{dt} = 2\pi(10)(8)(3) + \pi(10)^2(-5) = -20\pi$$

The volume is decreasing at the rate of about 62.8in.^{3}/min.

Question.17

If y is a differentiable function of x such that

$$\sin(x+y) + \cos(x-y) = y$$

find $\frac{dy}{dx}$.
Solution: Let $F(x,y) = \sin(x+y) + \cos(x-y) - y$, so that $F(x,y) = 0$. Then

$$F_x = \cos(x+y) - \sin(x-y)$$
$$F_y = \cos(x+y) - \sin(x-y)(-1) - 1$$

so

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-[\cos(x+y) - \sin(x-y)]}{\cos(x+y) + \sin(x-y) - 1}$$

Question.18

Second derivative of a function of two variables

Let $z = f(x, y)$, where $x = at$ and $y = bt$ for constants a and b. Assuming all necessary differentiability, find d^2z/dt^2 in terms of the partial derivatives of z.

Solution: By using the chain rule, we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

and

$$\begin{aligned}\frac{d^2z}{dt^2} &= \frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{d}{dt}\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\\ &= \frac{\partial z}{\partial x}\left[\frac{d}{dt}\left(\frac{dx}{dt}\right)\right] + \left[\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)\right]\frac{dx}{dt} + \frac{\partial z}{\partial y}\left[\frac{d}{dt}\left(\frac{dy}{dt}\right)\right] + \left[\frac{d}{dt}\left(\frac{\partial z}{\partial v}\right)\right]\frac{dy}{dt}\\ &= \left[\frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{dx}{dt}\left(\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial x \partial x}\frac{dy}{dt}\right)\right] + \left[\frac{\partial z}{\partial y}\frac{d^2y}{dt^2} + \frac{dy}{dt}\left(\frac{\partial^2 z}{\partial z \partial y}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}\right)\right]\end{aligned}$$

Substituting $\frac{dx}{dt} = a$ and $\frac{dy}{dt} = b$, we obtatin

$$\begin{aligned}\frac{d^2z}{dt^2} &= \left[\frac{\partial z}{\partial x}(0) + a\left(\frac{\partial^2 z}{\partial x^2}a + \frac{\partial^2 z}{\partial x \partial y}b\right)\right] + \left[\frac{\partial z}{\partial y}(0) + b\left(\frac{\partial^2 z}{\partial y \partial x}a + \frac{\partial^2 z}{\partial y^2}b\right)\right]\\ &= a^2\frac{\partial^2 z}{\partial x^2} + 2ab\frac{\partial^2 z}{\partial x \partial y} + b^2\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Question.19

Chain rule for two independent parameters

Let $z = 4x - y^2$, where $x = uv^2$ and $y = u^3v$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Solution: First find the partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(4x - y^2\right) = 4 \quad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(4x - y^2\right) = -2y$$
$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(uv^2\right) = v^2 \quad \frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(u^3v\right) = 3u^2v$$

and

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(uv^2\right) = 2uv \quad \frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left(u^3v\right) = u^3$$

Therefore, the chain rule for two independent parameters gives

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= (4)\left(v^2\right) + (-2y)\left(3u^2v\right) = 4v^2 - 2\left(u^3v\right)\left(3u^2v\right) = 4v^2 - 6u^5v^2\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= (4)(2uv) + (-2y)\left(u^3\right) = 8uv - 2\left(u^3v\right)u^3 = 8uv - 2u^6v\end{aligned}$$

Question.20

Finding a directional derivative using partial derivatives

Find the directional derivative of $f(x,y) = 3 - 2x^2 + y^3$ at the point $P(1,2)$ in the direction of the unit vector $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$.

Solution : First note that given vector $\mathbf{u}$ is indeed a unit vector. Then find the partial derivatives $f_x(x,y) = -4x$ and $f_x(x,y) = 3y^2$.

Then, since $u_1 = \frac{1}{2}$ and $u_2 = -\frac{\sqrt{3}}{2}$, we have

$$\begin{aligned}D_{\mathbf{u}}f(1,2) &= f_x(1,2)\left(\frac{1}{2}\right) + f_y(1,2)\left(-\frac{\sqrt{3}}{2}\right) \\ &= -4(1)\left(\frac{1}{2}\right) + 3(2)^2\left(-\frac{\sqrt{3}}{2}\right) = -2 - 6\sqrt{3} \approx -12.4\end{aligned}$$

Multiple Choice Question

Question.21

Compute the partial derivative of the function

$$f(x,y,z) = e^{1-x\cos(y)} + z\,e^{-1/(1+y^2)}$$

with respect to x at the point $(1,\ 0,\ \pi)$.

(a) -1

(b) $-1/e$

(c) 0

(d) π/e

(e) π

Answer We compute

$$\frac{\partial f}{\partial x} = e^{1-x\cos y}(-\cos y)$$

$$\frac{\partial f}{\partial x}\Big|_{(1,0,\pi)} = e^{1-1\cos 0}(-\cos 0) = e^{0}(-1) = -1.$$

The correct answer is (a).

Question.22

The maximum value of $(xy)^6$ on the ellipse $\frac{x^2}{4} + y^2 = 1$ occurs at a point $(x,\ y)$ for which y^2 is equal to

(a) $\sqrt{2}/3$

(b) $1/2$

(c) $2/3$

(d) $5/11$

(e) $10/11$

Answer . We want to maximize $f(x, y) = (xy)^6$ subject to the constraint $g(x, y) = x^2/4 + y^2 - 1 = 0$. The maximum occurs when

$$\nabla f = \lambda \nabla g,$$
$$g = 0.$$

We compute $\nabla f = 6x^5y^6\,\hat{\imath} + 6x^6y^5\,\hat{\jmath}$ and $\nabla g = (x/2)\,\hat{\imath} + 2y\,\hat{\jmath}$. So we must solve

$$6x^5y^6 = \frac{\lambda x}{2}$$
$$6x^6y^5 = 2\lambda y$$
$$\frac{x^2}{4} + y^2 = 1$$

Question.23

Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}.$$

(a) -1

(b) 0

(c) $\frac{1}{2}$

(d) 1

(e) the limit does not exist

Answer Along the line $y = kx$ with $k \neq 0$ the limit equals to

$$\lim_{x\to 0} \frac{kx^2}{x^2 + k^2x^2} = \frac{k}{1 + k^2}.$$

Different k's give rise to different limits and so the limit does not exist. The correct answer is (e).

Question.24

I see an object 3 miles to the East and 4 miles North. It appears to be moving at 1 mile per minute in the Southwest direction. At how many miles per minute is it getting closer to me?

(a) $\frac{1}{5\sqrt{2}}$

(b) $\frac{1}{5}$

Ar.swer I stand at (0, 0). The nbject starts at (3, 4) and trax•els in the direction of x #. Set f-(f) denote the position of the object at time t. Then f (t) == r V t where r o is thy initial position vectnr and V is the constant velocity vector of the object. So r o ' 311 4# and Y o is a vector in the direction of i # o•ith magnitude 1 (== the speed of the objec thus

.

$$\vec{r}(t) = \left(3 - \frac{t}{\sqrt{2}}\right)\mathbf{i} + \left(4 - \frac{t}{\sqrt{2}}\right)\mathbf{j}.$$

The distance betv•een the object and me is (I) | and so the speed at which the ob jest is getting closer to rue is

$$\frac{d|\vec{r}(t)|}{dt},$$

K'e compute

$$|\vec{r}(t)| = \sqrt{\left(3 - \frac{t}{\sqrt{2}}\right)^2 + \left(4 - \frac{t}{\sqrt{2}}\right)^2}$$

$$= \sqrt{t^2 - 7\sqrt{2}t + 25}.$$

Differentiating with respect to P we get

$$\frac{d|\vec{r}(t)|}{dt} = \frac{2t - 7\sqrt{2}}{2\sqrt{t^2 - 7\sqrt{2}t + 25}}$$

and so

$$-\frac{d|\vec{r}(0)|}{dt} = \frac{-7\sqrt{2}}{\ldots} = \frac{7}{\ldots}$$

The correct answer is (c). Q

Question.25

The tangent plane to the graph of the function $z = x^2y + 1/(1 + y^2)$ at the point $(1, 1, 3/2)$ contains point $(2, 2, t)$ for which value of t?

(a) $8\frac{1}{5}$

(b) $1 + \frac{7}{4}\sqrt{2}$

(c) 4

(d) 5

(e) none of the above

Answer The equation of the surface is $f(x, y, z) = x^2y + 1/(1 + y^2) - z$ and so the equation of the tangent plane at $P_0 = (1, 1, 3/2)$ is

$$f_x(P_0) \cdot (x-1) + f_y(P_0) \cdot (y-1) - \left(z - \frac{3}{2}\right) = 0.$$

Computing the partials we get $f_x = 2xy$ and $f_y = x^2 - 2y/(1 + y^2)^2$. Hence $f_x(P_0) = 2$ and $f_y(P_0) = 1/2$ and so the equation of the tangent plane is

$$2x + \frac{y}{2} - z = 1.$$

At $(2, 2, t)$ this gives $4 + 1 - t = 1$, i.e. $t = 4$. The correct answer is (c).

Question.26

A particle moves in a circle according to the equations $\vec{r}(t) = \cos(t^2)\hat{\imath} + \sin(t^2)\hat{\jmath}$.

The magnitude of the normal component of the acceleration at time t is

(a) 0

(b) 1

(c) 6t

(d) t^2

(e) $4t^2$

Answer Recall that in the $\vec{T}\,\vec{N}$ frame the acceleration decomposes as $\vec{a} = a_T\vec{T} + a_N\vec{N}$. We want to find the value of a_N. We also have

$$a_T = \frac{d|\vec{v}|}{dt} \qquad a_N = \sqrt{|\vec{a}|^2 - a_T^2},$$

and so we have to compute $|\vec{a}|^2$ and a_T. To determine $\vec{a}$ we differentiate $\vec{r}(t)$:

$$\vec{v} = \vec{r}'(t) = -2t\sin(t^2)\hat{\imath} + 2t\cos(t^2)\hat{\jmath},$$

$$\vec{a} = \vec{r}''(t) = (2\cos(t^2) - 4t^2\sin(t^2))\hat{\imath} - (2\sin(t^2) + 4t^2\cos(t^2))\hat{\jmath}.$$

Using this we compute $|\vec{v}| = 2t$ and $a_T = (2t)' = 2$. Similarly we have $|\vec{a}|^2 = 4 + 16t^2$ and so

$$a_N = \sqrt{4 + 16t^2 - 4} = 4t.$$

Correct answer is (e)

Question.27

If $g(x, y) = x^3y^2 - y$, then what is $g_{yx}(2, 3)$?

(a) 54

(b) 107

(c) 108

(d) 72

(e) 71

Correct answer is (d)

Question.28

The position of a particle is $\mathbf{r}(t) = 2e^{2t}\,\mathbf{i} + 3t^2\,\mathbf{j}$. What is the acceleration at $t = 0$?

(a) $2\mathbf{i}$

(b) $2\mathbf{i} + 6\mathbf{j}$

(c) $8e\mathbf{i}$

(d) $6\mathbf{j}$

Correct answer is (b)

Question.29

Find an equation of the tangent plane to the surface $f(x, y, z) = 10$ at the point $(2, 3, 6)$.

(a) $2(x-4) + 3(y-3) + 6(z-12) = 0$

(b) $2(x-4) + 3(y-3) + 6(z-12) = 10$

(c) $4(x-2) + 3(y-3) + 12(z-6) = 10$

(d) $4(x-2) + 3(y-3) + 12(z-6) = 0$

(e) $2x + 3y + 6z = 49$

Correct answer is (b)

Question.30

What is the value of the iterated integral

$$\int_0^{\pi} \int_0^{2} (2r + 2r\theta)\, dr\, d\theta\,?$$

(a) $4\pi^2/3$

(b) $4\pi + 2\pi^2$

(c) $2\pi + \pi^2$

(d) $\pi + 4\pi^2$

(e) $8\pi + 4\pi^2$

Correct answer is (b)