Group theory

Problem. 1 Let a ∈ G for some group. Show that a and a^{-1} have the same order

Solution. 1 Suppose a k = e for some integer k. Multiplying both sides with a^{-k} gives e = a the smallest positive k making the left hand side true, must also be the smallest k making the right hand side true . In other words, a k = e ⇐⇒ a −k = e. It follows that

Problem. 2 Let a, b ∈ G. Show that ab and ba have the same order.

Solution. 2 Let k be an integer such that (ab) k = e. That is, (ab) k = abab · · · ab = e Multiply from the left with a^{-1} and then from the right with $b^{-1}a^{-1} \cdots b^{-1}$. This gives by using that $(ab)^{-1} = b^{-1}a^{-1}$. e = $a^{-1} b^{-1} \cdots b^{-1} = (ba)^{-1.}$ In other words $(ba)^{-1}$ also have the property that when it is raised to the k^{th} power, it becomes the identity. In fact, we have the equality (ab) k = e ⇐⇒ (ba) −k = e. Since $(ba)^{-1}$ and ba have the same order, we are now done.

Problem. 3 Suppose ab = ba, and that $a^m = b^n = e$. Show that $(ab)^{mn} = e$

Solution. 3 We have that $(ab)^{mn} = (ab)(ab)\cdots(ab) \mid \{z\}$ mn times $= a^{mn}b^{mn}$ Now, $a^{mn} = (a^m)^n$ and $b^{mn} = (b^n)^m$, so both these are the identity element. Hence $a^{mn}b^{mn} = e^2 = e$ and we are done.

Problem. 4 Let a be an element in a group G, such that a has order 18. What order does a 2 have? What is the inverse of a 9 ?

Solution.4 Suppose that a^2 has order k. Then a 2k = e, so in particular k ≥ 9. On the other hand, $(a^2)^9 =$

$a^{18} = e$, so the order of a 2 is at most 9. Therefore, it must be equal to 9. We know that $(a^9)^2 = e$, so a 9 is its own inverse.

Problem. 5 Let G be a group with a and b in G. Show the equivalence $a^k = e \iff (bab^{-1})^k == e$.

Can you draw any conclusion regarding order (a) and order (bab^{-1})

Solution. 5 We show the equivalence as follows: $a^k = e \iff ba^k b^{-1} = beb^{-1}$

$\iff b\,aa \cdots a\, b^{-1} \mid \{z\}\ k = e.$

The first step is done by multiplying on the left with b and on the right with b^{-1} , and then we just rewrite and expand. Subsequent manipulation gives

$ak = e \iff ba(b^{-1} b)a(b^{-1} b)a \cdots a(b^{-1} b)ab^{-1} = e$

$\iff (bab^{-1})(bab^{-1})\cdots(bab^{-1}) = e$

$\Leftarrow\Rightarrow (bab^{-1})\, k = e.$

The first step here is done by inserting $e = b^{-1} b$ between a's. As multiplication by e does not change the expression, this step is valid. The second step is just regrouping. If we are really picky, we can say that we use the associativity of the group operation. This result allow us to use the same reasoning as in earlier exercises, and conclude that a and bab^{-1} have the same order.

Problem. 6 Let G be a group, and let a, b be in G. Define the se H_b: = $\{bab^{-1} : a \in G\}$ Prove that H_b is a subgroup of G.

Solution. 6 It is enough to show that H_b is closed under multiplication and taking inverses.

Closeness under inverses is straightforward: We note that if $bab^{-1} \in H_b$, then $ba^{-1} b^{-1} \in H_b$ is in H as well. Furthermore, $(bab^{-1})(ba^{-1} b^{-1}) = ba(b^{-1} b)a^{-1} b^{-1} = b(aa^{-1})b^{-1} = e$, so the inverse of bab^{-1} is given by $ba^{-1} b^{-1}$.

Closeness under multiplication follows a similar pattern: Suppose ba^1b^{-1} and ba^2b^{-1} are elements in H_b. Then their product $(ba^1b^{-1})(ba^2b^{-1}) = b(a^1a^2)b^{-1}$ must also be in H_b, since a^1a^2 is an element in G.

Problem. 7The set H = {0, 3, 6, 9 } is a subgroup of Z12. Find all cosets of H

Solution. 7The cosets are produced by multiplying H (on the right) with elements in Z_{12}. Group multiplication in Z_{12} is addition mod 12. The three cosets are {0, 3, 6, 9} By adding 0, 3, 6 or 9 to H {1, 4, 7, 10} By adding 1, 4, 7 or 10 to H {2, 5, 8, 11} By adding 2, 5, 8 or 11 to H. Note that the union of the cosets give the entire group Z_{12}.

Problem. 8Find the order of all elements and all subgroups of Z_4

Solution. 8 Case by case checking shows that order(0) = 1, order(1) = order(3) = 4, order(2) = 2. Since Z_4 is a cyclic group — it has 1 as a generator, all subgroups are also cyclic. re also cyclic. It is therefore enough to see what group each element generates. Since 1 and 3 have order 4, these generate the entire group. The only non-trivial4 subgroup is therefore the one generated by 2, namely {0, 2}. The subgroups are therefore {e}, {0, 2}, Z_4.

Problem. 9 Suppose $G = \{g_1, \ldots, g_n\}$ is a finite Abelian group and let $c = g_1 g_2 \cdots g_n$. Prove that $g_2 = e$

Solution. 9 Since G is Abelian, $c_2 = g_1 g_1 g_2 g_2 \cdots g_n g_n$, and we can rearrange all factors as we wish. For each factor $g_i g_i$, there are two cases to consider: Either $g_i g_i = e$, in the case gi is its own inverse, or g_i has some other inverse, g_j. In this case, we can rearrange these two pairs such that we have

$(g_i g_k)(g_i g_k) = e^2 = e$. Every element is therefore canceled by some other element in the big product, and the result is the identity element e.

Problem. 10 How many subgroups does Zn have, if $n = 2^4 \times 3^2 \times 5$

Solution. 10 As we saw in the previous exercise, if k|n, then kZ_n/k is a subgroup of Zn. Furthermore, we can only have cyclic subgroups and every cyclic subgroup is of this form. In other words H is a subgroup of Zn

$\Leftarrow\Rightarrow$ $H = kZ_n/k$ for some k dividing n. It suffices to compute the number of divisors of n. We have $5 \times 3 \times 2 = 30$ divisors, as for each prime number p in the factorization of n, we must choose how many times it appear in a divisor.

Problem. 11 Determine which of the following statements are true. By true, we mean always true.

(a) If a, b be elements in a group such that order(a) = 4 and order(b) = 2 then order(ab) = 8.

(b) The complex number $e^{2\pi i/n}$ generates a cyclic group of size n in $\langle C \setminus \{0\}, \times \rangle$.

(c) If |G| = m and |H| = n then G × H has a subgroup of size m

Solution. 11 (a) No, this statement is false. Take for example a = 1 and b = 2 in $(Z_4, +)$.

(b) Yes, this is true. One can easily check that if $\xi = e^{2\pi i/n}$ then $H = \{1, \xi, \xi^2, \xi^3, \ldots, \xi^{n-1}\}$ is a set of n different numbers, it is generated by ξ, and $\xi^n = 1$. In fact, the elements in H are the n different complex solutions to $x^n - 1 = 0$. Thus, H is a cyclic subgroup.

(c) Yes, this is true — consider all elements of the form $K = \{(g, e) : g \in G\}$. Then |K| = |G| and it is a routine exercise to show that K is indeed a subgroup.

Problem. 12 Consider the following multiplication table for a group G and solve the following problems. ◦ e a b c d f

e e a b c d f

a a e d f b c

b b c e a f d

c c b f d e a

d d f a e c b

f f d c b a e

(1) (a) Determine if G is commutative.

(b) Determine if G is cyclic.

(c) Find the inverse of d.

(d) Find all subgroups of size 2.

Solution. 13 We have the following:

(a) G is not commutative, because $b \circ a = c$, but $a \circ b = d$.

(b) We just concluded that G is not commutative, so it cannot be cyclic.

(c) We look in the row of d, and see that $d \circ c = e$, so $d^{-1} = c$.

(d) Any subgroup of size 2 must be of the form {e, x}, where $x^2 = e$. We look in the table, and see that

$a^2 = b^2 = f^2 = e$, so {e, a}, {e, b} and {e, f} are all subgroups of size 2.

(e) There are no subgroups of size 4 since that would violate Lagrange's theorem.

(f) A subgroup of size 3 must be cyclic since 3 is a prime. The table gives that $c \circ c = d$ and that $c \circ d = a$. Therefore, $c \circ c \circ c = e$, and c is an element of order 3. It follows that {e, c, d} is a subgroup of order 3.

(g) We want to find x such that axc = f. Solving for x by multiplying with appropriate inverses gives that $x = a^{-1} f c^{-1}$. From the table we see that $a^{-1} = a$, $c^{-1} = d$. Thus, x = af d. We can then read off that $a \circ f = c$, so $x = c \circ d = e$. (e) Are there any subgroups of size 4?

(f) Find a subgroup of size 3.

(g) Find an element x such that axc = f.

Problem:14 Suppose $\pi \in Sn$. Show that $inv(rev(\pi)) = \binom{n}{2} - inv(\pi)$.

Solution:14 If the entries a and b form an inversion in π, they do not form an inversion in rev(π) and vice versa. In other words, every pair of entries, (a, b) is an inversion in exactly one of π and rev(π). Since the total number of such pairs for a permutation in S_n is $\binom{n}{2}$, we must have that $\binom{n}{2} = inv(rev(\pi)) + inv(\pi)$. This proves the result

Problem.15 How many permutations in S_{10} are there of type (2, 2, 2, 2, 1, 1)?

Solution. 15

We need to count all possible ways to construct four 2-cycles. Choosing the elements to be in the two-cycles can be done in $\binom{10}{2,2,2,2,1,1}$

ways, but we need to divide by 4! because the order of the two-cycles
does not matter. The answer is therefore
$\frac{10!}{2^4 \times 4!}$

Problem. 16 Let S_8 be the group of permutations on 8 elements.
Solution.16 Describe an Abelian subgroup of S_8 with 10 elements. It would be convenient to look for a cyclic subgroup with 10 elements, since all cyclic groups are Abelian. Every cyclic (sub)group has a generator of order 10, so we need to find a permutation in S_8 with order 10. The order of a permutation is determined by the lcm of the lengths of the cycles. We cannot fit a single cycle of length 10,but we can find a permutation with a cycle of length 5, and a cycle of length 2. For example , pi_ = (12345)(67)(8)
is in S_8 and has order 10. It follows that (pi) the cyclic group
generated by *pi* is an Abelian subgroup of size 10.

Problem. 17
Suppose $ab = ba$, and that $a^m = b^n = e$. Show that $(ab)^{mn} = e$.
Solution. 17 We have that
$(ab)^{mn} = (ab)(ab) \cdots (ab) \mid \{z\}_{mn \text{ times}} = a^{mn}b^{mn}$
Now,
$a^{mn} = (a^m)^n$ and $b^{mn} = (b^n)^m$,
so both these are the identity
element. Hence $a^{mn}b^{mn} = e^2 = e$ and we are done.

Problem. 18
Find the order of all elements, and all subgroups of $\mathbf{Z}_2$ **multiplication modulo** $\mathbf{Z}_2$.
Solution. 18
The elements are $e = (0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$, and group
operation is given by element-wise addition mod 2.It is straightforward to verify that $(0, 0)$ has order 1, and all otherelements have order 2.Since the group has size 4, subgroups can only have 1, 2 or 4elements. The subgroups are$\{e\}$, $\{e, (0, 1)\}$, $\{e, (1, 0)\}$, $\{e, (1, 1)\}$ and $\mathbf{Z}_2\mathbf{Z}_2$.
The identity element is always the only possible subgroup with onlyone element.Subgroups of size 2 must have e and one additional element, and by checking, all three other elements generate a subgroup of size 2.

Problem. 19
How many permutations in S_{12} are there of type $(3, 3, 3, 3)$?

Solution. 19
We need to partition the permutation into 4 3-cycles. Choosing the elements to be in the 3-cycles can be done in $\binom{12}{3,3,3,3}$ ways, butwe need to divide by 4! because the order of the 3-cycles does not matter. Furthermore, the elements in each 3-cycle can be ordered in two different ways. We therefore have 2 choices for each cycle.
$\frac{12! \times 2^4}{(3!)^4 \times 4!}$

Problem. 20
Consider the following multiplication table for a group G and answer
the following questions.

×	a	b	c	d	f
a	b	f	d	a	c
b	f	c	a	b	d
c	d	a	f	c	b
d	a	b	c	d	f
f	c	d	b	f	a

(2)
(a) Which element is the identity element?
(b) Is the group commutative?

(c) Is there some x belongs to G such that $x_3 = d$?

Solution. 20
(a) From the table, we see from row d that multiplication with d has no effect, so d is the identity element.
(b) The group is commutative, since the table is the same under transposition (seen as a matrix).
(c) We know that d is the identity element, so the question if there is some x with order 3. This is not possible in a group with 5 elements.

1. **This is an abelian group { – 3 n : n ε Z } under?**
   A. division
   B. subtraction
   C. addition
   D. multiplication
   Answer-(c)
2. **What is the inverse of – ı If G = { 1, -1, ı, – ı } is group under multiplication?**
   A. −1
   B. ı
   C. 1
   3. The monoid is a?
   A. a non-abelian group
   B. groupoid
   C. A group
   D. a commutative groupD. None of Above
   Answer-(b)

4. The monoid is a?
A. a non-abelian group
B. groupoid
C. A group
D. a commutative group

Answer-(d)

5. **3. The monoid is a?**
A. a non-abelian group
B. groupoid
C. A group
D. a commutative group

Answer-c

6. What is the value of (a- 1 b)- 1 is in the group (G, .)?
A. b- 1a
B. ab-1
C. ba-1
D. a-1b

Answer-(a)

7. What is the inverse of an if (Z,*) is a group with a*b = a+b+1 ∀ a, b ∈Z?
A. -2
B. 0
C. -a-2
D. a-2

Answer-©

8. Which sentence is true?
A. Set of all matrices forms a group under multiplication
B. Set of all rational negative numbers forms a group under multiplication
C. Set of all non-singular matrices forms a group under multiplication
D. Both (b) and (c)

Answer-(c)

9. Which statement is false?
A. The set of rational integers is an abelian group under addition
B. The set of rational numbers form an abelian group under multiplication
C. The set of rational numbers is an abelian group under addition
D. None of these

Answer-(b)

10. What is the identity element In the group G = {2, 4, 6, 8) under multiplication modulo 10?
A. 5
B. 9
C. 6
D. 12

Answer-(c)