Differential Equation

MULTIPLE CHOICE QUESTIONS -

1. What is the order of the given Differential Equation ?

$$\frac{d^3y}{dx^3} + \mathbf{3x}\frac{dy}{dx} = e^y$$

(a) 0 (b) 1 (c) 2 (d) 3

[Ans : (d)]

2. What is the degree of the given Differential Equation?

$$\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^{\mathbf{2}} - \mathbf{3y}\frac{dy}{dx} + \boldsymbol{y} = \mathbf{0}$$

(a) 0 (b) 1 (c) 2 (d) 3

[Ans : (b)]

3. Determine the value of the constant r for which $y = e^{rx}$ is a solution of the following equation - ***3y' = 2y.***

(a) r = 0 (b) r = 1/3 (c) r = 2/3 (d) r = 1

[Ans : (c)]

4. Find a function y = f(x) satisfying the given differential equation and the prescribed initial condition –

dx = cos 2x ; y(0) = 1

(a) $y(x) = \frac{1}{2}\sin 2x + 1$

(b) $y(x) = \sin 2x + 1$

(c) $y(x) = \frac{1}{2}\sin 2x - 1$

(d) $y(x) = \sin 2x - 1$

[Ans : (a)]

5. Find the integrating factor of the following equation –

$$(3y + 4xy^2)dx + (2x + 3x^2 y)dy = 0$$

(a)xy (b)x^2y (c)xy^2 (d) y

[Ans: (b)]

6. In Lotka-Volterra Model of Predator-Prey population what happens to the Prey in the absence of any Predator?

(a) The population of prey will increase exponentially
(b) The population of prey will decrease exponentially
(c) The population of prey remains constant
(d) None of the above

[Ans : (a)]

7. Which period is longer? Incubation period or latent period?

(a) Incubation Period
(b) Latent Period
(c) Depends on various factors
(d) Both remain same

[Ans : (a)]

8. The differential equation $\mathbf{2\frac{dy}{dx} + x^2y = 2x + 3, y(0) = 5}$ is -

(a) linear
(b) nonlinear
(c) linear with fixed constants
(d) undeterminable to be linear or nonlinear

[Ans : (a)]

9. A differential equation is considered to be ordinary if it has

(a)one dependent variable

(b) more than one dependent variable

(c) one independent variable

(d)more than one independent variable

[Ans : (c)]

10. The form of the exact solution to $2\frac{dy}{dx} + 3y = e^{-x}, y(0) = 5$ is –

(a) $Ae^{-1.5x} + Be^{-x}$

(b) $Ae^{-1.5x} + Bxe^{-x}$

(c) $Ae^{1.5x} + Be^{-x}$

(d) $Ae^{1.5x} + Bxe^{-x}$

[Ans : (a)]

SUBJECTIVE QUESTIONS

11.Explain Newton's law of cooling and translate it into a differential equation.

Newton's law of cooling may be stated in this way: The *time rate of change* (the rate of change with respect to time t) of the temperature *T(t)* of a body is proportional to the difference between *T* and the temperature *A* of the surrounding medium. That is,

$$\frac{dT}{dt} = -k\,(T - A)$$

Where *k* is a positive constant. If $T > A$, then $dT/dt < 0$, so the temperature is a decreasing function of *t* and the body is cooling. But if $T < A$, then $dT/dt > 0$, so that *T* is increasing.

12.A specimen of charcoal found at Stonehenge turns out to contain 63% as much as ^{14}C as a sample of present day charcoal of equal mass. What is the age of the sample ?

We take $t = 0$ as the time of the death of the tree from which the Stonehenge charcoal was made and N_0 as the no. of ^{14}C atoms that the sample contained then. We are given that
$N = (0.63)N_0$ now, so we solve the equation $(0.63)N_0 = N_0 e^{-kt}$ with the value $k = 0.0001216$.
Thus we find that

$$t = \frac{\ln(0.63)}{0.0001216} = 3800 yrs\ (approx.)$$

Thus the sample is about 3800 years old.

13. The following nonlinear differential equation can be solved exactly by separation of variables, then the value of $\theta(100)$ most nearly is -

$$\frac{d\theta}{dt} = -10^{-6}(\theta^2 - 81), \theta(0) = 1000$$

Solution :

$$\frac{dB}{dt} = -10^{-6}\left(\theta^2 - 81\right)$$

$$\left(\frac{dB}{B^2 - 81} --- 10'' dl \right.$$

$$\text{J} \quad \frac{}{(B- \quad 8+ \quad g)'} \quad -)\, IO-^{6}\, dr$$

$$\frac{1}{(\phi - 9\breve{g}/?+9)} = \frac{A}{6-9} + \frac{B}{6+9}$$

$$1 = \quad (\phi + 9\,) + B(B - 9$$

$$1 = (\quad + \quad \emptyset + (9A - \; 9B\,)$$

$$A + B -- 0$$

$$9A - 9B \quad --1$$

$$A -\!\!\cdot\; \frac{1}{18}$$

$$B --- \frac{1}{18}$$

$$\int\left(\frac{1}{18}\right)\left(\frac{1}{8-9}\right)d8 - \text{J}\left(\frac{\text{I}}{18}\right)\left(\frac{\text{I}}{8+9}\right. \quad d8 --- J\,10 \quad dt$$

$$\frac{!\text{n}(a-9)-}{18} \quad \frac{\text{l«}(a+ \quad)-}{18} \quad -10\text{-'I} + c$$

$$\frac{1}{18}\ln\left(\frac{Q-9}{Q+9}\right) = - \text{ t0'''i} + \text{c}$$

To find the value of c, 8(0) = 1000

$$\frac{\acute{\imath}}{18}\ \text{Jq}\left(\frac{1000-9}{1000+9}\right. \quad =\text{-lO''xO+r}$$

$$\text{r} = -0.0010000$$

Substituting the value of c, we get

$$\frac{1}{18}\ln\left(\frac{\theta-9}{\theta+9}\right)=-10^{-6}t-0.0010000$$

To find the value of θ at $t=100$,

$$\frac{1}{18}\ln\left(\frac{\theta-9}{\theta+9}\right)=-10^{-6}(100)-0.0010000$$

$$\ln\left(\frac{\theta-9}{\theta+9}\right)=-0.0198$$

$$e^{\ln\left(\frac{\theta-9}{\theta+9}\right)}=e^{-0.0198}$$

$$\frac{\theta-9}{\theta+9}=0.98039$$

$$\theta=909.10$$

14. If an archaeologist uncovers a sea shell which contains 60% of the living ^{14}C of a living shell. How old do you estimate that shell and thus that site to be? (half-life of ^{14}C is 5568 years)

Solution :

τ = 5568 years

Let the initial quantity be 100x = x_0

$t_0 = 0$, $x_0 = 100x$

t = ? , x(t) = 60x

$$x = x_0 e^{-k(t-t_0)}$$

$$60x = 100x \times e^{-k(t-0)}$$

$$\frac{3}{5} = e^{-kt}$$

Using formula for half life, we have –

$$k = \frac{\ln 2}{\tau}$$

$$\ln\left|\frac{3}{5}\right| = -kt$$

$$\ln\left|\frac{3}{5}\right| = \frac{\ln 2}{\tau}t$$

$$\ln\left|\frac{3}{5}\right| = \frac{\ln 2}{5568}t$$

Solving this equation, we get t = 4103 years.

Thus, the site is about 4103 years old.

15. What are Bernoulli Equations ?

A first order differential equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$$

is called a Bernoulli Equation. If either n = 0 or n = 1, then the equation is linear, otherwise the substitution

$$v = y^{1-n}$$

the equation into the linear equation.

$$\frac{dv}{dx} + (1-n)\,P(x)\,v = (1-n)\,Q(x)$$

16. Find a particular solution of $y'' - 4y = 2e^{3x}$.

Any derivative of e3x is a constant multiple of e^{3x}, so it is reasonable to try

$$y_p(x) = Ae^{3x}$$

Then $y''_p = 9Ae^{3x}$,so the given differential equation will be satisfied provided that

$$9Ae^{3x} - 4(Ae^{3x}) = 2Ae^{3x} ;$$

That is 5A = 2, so that A = 2/5 . Thus, our particular solution is $y_p(x) = 2/5e^{3x}$.

17. Solve $(3x^2 y^3 – 5x^4) dx + (y + 3x^3 y^2) dy = 0$

Solution :

In this case

$M(x , y) = 3x^2 y^3 – 5x^4$

$N(x , y) = y + 3x^3 y^2$

So our equation is exact.

$$= \int M(x, y)dx$$
$$= \int (3x^2 y^3 - 5x^4)dx$$
$$= x^3 y^3 - x^5 + f(x)$$

$$3x^3y^2 + \frac{dF}{dy} = y + 3x^3y^2$$

$$\frac{dF}{dy} = y$$

$$f(g) = \frac{y^2}{2} + C_1$$

$$I(x, y) = x^3y^3 - x^5 + \frac{y^2}{2} + C_1$$

$$x^3y^3 - x^5 + \frac{y^2}{2} = C$$

18. Find a general solution of the fifth order differential equation

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$$

Solution :

The characteristic equation is

$$9r^5 - 6r^4 + r^3 = r^3(9r^2 - 6r + 1) = r^3(3r-1)^2 = 0$$

It has a triple root r = 0 and the double root r = ⅓. The triple root r = 0 contributes

$$C_1e^{0.x} + c_2xe^{0.x} + c_3x^2e^{0.x} = c_1 + c_2x + c_3x^2$$

To the solution. While the double root r =⅓ contributes $c_4e^{x/3} + c_5xe^{x/3}$. Hence a general solution of the given differential equation is

$$y(x) = c_1 + c_2x + c_3x^2 + c_4e^{x/3} + c_5xe^{x/3}.$$

19. Explain linear dependence of two functions

Solution :

Two functions are said to be linearly dependent on an open interval provided that they are not linearly independent there ,that is one of them is a constant multiple of other. We can always determine whether 2 given functions *f* and *g* are linearly dependent on a given interval *I* by noting at a glance whether either of the two quotients *f/g* or *g/f* is a constant valued function on *I.*

20. Explain Torricelli's law and convert it into a differential equation.

Solution :

Torricelli's law implies that the *time rate of change* of the volume *V* of water in a draining tank is proportional to the square root of the depth *y* of water in the tank :

$$\frac{dV}{dt} = -ky^{\frac{1}{2}}$$

Where *k* is a constant. If the tank is a cylinder with vertical sides and cross-sectional area *A*, then *V* = *Ay*, so *dV/dt* = *A*. (*dy/dt*). So the equation will take the form

$$\frac{dy}{dt} = -hy^{\frac{1}{2}}$$

where *h* = *k/A* is a constant.

21. Solve the following differential equation :

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0.$$

Solution :

Let $M(x, y) = 6xy - y^3$ and $N(x, y) = 4y + 3x^2 - 3xy^2$. The given equation is exact because

$$\frac{\partial M}{\partial y} = 6x - 3y^2 = \frac{\partial N}{\partial x}.$$

Integrating $\partial F/\partial x = M(x, y)$ with respect to x, we get

$$F(x, y) = \int (6xy - y^3)\,dx = 3x^2y - xy^3 + g(y).$$

Then we differentiate with respect to y and set $\partial F/\partial y = N(x, y)$. This yields

$$\frac{\partial F}{\partial y} = 3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2,$$

and it follows that $g'(y) = 4y$. Hence $g(y) = 2y^2 + C_1$, and thus

$$F(x, y) = 3x^2y - xy^3 + 2y^2 + C_1.$$

Therefore, a general solution of the differential equation is defined implicitly by the equation

$$3x^2y - xy^3 + 2y^2 = C$$

22.

Verify that the function $y(x) = 2x^{1/2} - x^{1/2}\ln x$ satisfies the differential equation

$$4x^2y'' + y = 0$$

for all $x > 0$.

Solution:

First we compute the derivatives

$$y'(x) = -\tfrac{1}{2}x^{-1/2}\ln x \quad \text{and} \quad y''(x) = \tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}.$$

Then substitution into Eq. (10) yields

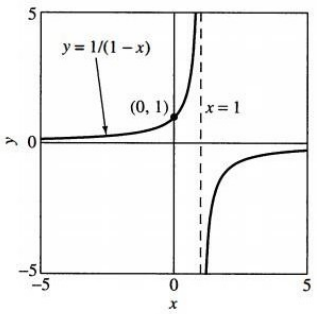
$$4x^2y'' + y = 4x^2\left(\tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}\right) + 2x^{1/2} - x^{1/2}\ln x = 0$$

if x is positive, so the differential equation is satisfied for all $x > 0$.

23.

Figure 1.1.5 shows the two "connected" branches of the graph $y = 1/(1-x)$. The left-hand branch is the graph of a (continuous) solution of the differential equation $y' = y^2$ that is defined on the interval $(-\infty, 1)$. The right-hand branch is the graph of a *different* solution of the differential equation that is defined (and continuous) on the different interval $(1, \infty)$. So the single formula $y(x) = 1/(1-x)$ actually defines two different solutions (with different domains of definition) of the same differential equation $y' = y^2$.

Solution:



If A and B are constants and

$$y(x) = A\cos 3x + B\sin 3x,$$

then two successive differentiations yield

$$y'(x) = -3A\sin 3x + 3B\cos 3x,$$
$$y''(x) = -9A\cos 3x - 9B\sin 3x = -9y(x)$$

for all x. Consequently, Eq. (14) defines what it is natural to call a *two-parameter family* of solutions of the second-order differential equation

$$y'' + 9y = 0$$

24.

Given the solution $y(x) = 1/(C - x)$ of the differential equation $dy/dx = y^2$ discussed in Example 7, solve the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(1) = 2.$$

Solution:

We need only find a value of C so that the solution $y(x) = 1/(C - x)$ satisfies the initial condition $y(1) = 2$. Substitution of the values $x = 1$ and $y = 2$ in the given solution yields

$$2 = y(1) = \frac{1}{C - 1},$$

so $2C - 2 = 1$, and hence $C = \frac{3}{2}$. With this value of C we obtain the desired solution

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$

25.

Solve the initial value problem

$$\frac{dy}{dx} = 2x + 3, \quad y(1) = 2.$$

Solution:

Integration of both sides of the differential equation as in Eq. (2) immediately yields the general solution

$$y(x) = \int (2x + 3)\,dx = x^2 + 3x + C.$$

Figure 1.2.3 shows the graph $y = x^2 + 3x + C$ for various values of C. The particular solution we seek corresponds to the curve that passes through the point $(1, 2)$, thereby satisfying the initial condition

$$y(1) = (1)^2 + 3 \cdot (1) + C = 2.$$

It follows that $C = -2$, so the desired particular solution is

$$y(x) = x^2 + 3x - 2.$$

26.

A lunar lander is falling freely toward the surface of the moon at a speed of 450 meters per second (m/s). Its retrorockets, when fired, provide a constant deceleration of 2.5 meters per second per second (m/s^2) (the gravitational acceleration produced by the moon is assumed to be included in the given deceleration). At what height above the lunar surface should the retrorockets be activated to ensure a "soft touchdown" ($v = 0$ at impact)?

Solution:

We denote by $x(t)$ the height of the lunar lander above the surface, as indicated in Fig. 1.2.4. We let $t = 0$ denote the time at which the retrorockets should be fired. Then $v_0 = -450$ (m/s, negative because the height $x(t)$ is decreasing), and $a = +2.5$, because an upward thrust increases the velocity v (although it decreases the *speed* $|v|$). Then Eqs. (10) and (11) become

$$v(t) = 2.5t - 450$$

and

$$x(t) = 1.25t^2 - 450t + x_0,$$

where x_0 is the height of the lander above the lunar surface at the time $t = 0$ when the retrorockets should be activated.

From Eq. (12) we see that $v = 0$ (soft touchdown) occurs when $t = 450/2.5 = 180$ s (that is, 3 minutes); then substitution of $t = 180$, $x = 0$ into Eq. (13) yields

$$x_0 = 0 - (1.25)(180)^2 + 450(180) = 40{,}500$$

meters—that is, $x_0 = 40.5$ km $\approx 25\frac{1}{6}$ miles. Thus the retrorockets should be activated when the lunar lander is 40.5 kilometers above the surface of the moon, and it will touch down softly on the lunar surface after 3 minutes of decelerating descent.

27.

Solve the differential equation

$$\frac{dy}{dx} = \frac{4 - 2x}{3y^2 - 5}.$$

Solution:

When we separate the variables and integrate both sides, we get

$$\int (3y^2 - 5)\,dy = \int (4 - 2x)\,dx;$$
$$y^3 - 5y = 4x - x^2 + C.$$

This equation is not readily solved for y as an explicit function of x.

28.

Find all solutions of the differential equation

$$\frac{dy}{dx} = 6x(y-1)^{2/3}.$$

Solution:

Separation of variables gives

$$\int \frac{1}{3(y-1)^{2/3}}\,dy = \int 2x\,dx;$$

$$(y-1)^{1/3} = x^2 + C;$$

$$y(x) = 1 + (x^2 + C)^3.$$

29.

According to data listed at **www.census.gov**, the world's total population reached 6 billion persons in mid-1999, and was then increasing at the rate of about 212 thousand persons each day. Assuming that natural population growth at this rate continues, we want to answer these questions:

(a) What is the annual growth rate k?

(b) What will be the world population at the middle of the 21st century?

(c) How long will it take the world population to increase tenfold—thereby reaching the 60 billion that some demographers believe to be the maximum for which the planet can provide adequate food supplies?

Solution:

(a) We measure the world population $P(t)$ in billions and measure time in years. We take $t = 0$ to correspond to (mid) 1999, so $P_0 = 6$. The fact that P is increasing by 212,000, or 0.000212 billion, persons per day at time $t = 0$ means that

$$P'(0) = (0.000212)(365.25) \approx 0.07743$$

billion per year. From the natural growth equation $P' = kP$ with $t = 0$ we now obtain

$$k = \frac{P'(0)}{P(0)} \approx \frac{0.07743}{6} \approx 0.0129.$$

Thus the world population was growing at the rate of about 1.29% annually in 1999. This value of k gives the world population function

$$P(t) = 6e^{0.0129t}.$$

(b) With $t = 51$ we obtain the prediction

$$P(51) = 6e^{(0.0129)(51)} \approx 11.58 \text{ (billion)}$$

for the world population in mid-2050 (so the population will almost have doubled in the just over a half-century since 1999).

(c) The world population should reach 60 billion when

$$60 = 6e^{0.0129t}; \quad \text{that is, when } t = \frac{\ln 10}{0.0129} \approx 178;$$

and thus in the year 2177.

30.

A specimen of charcoal found at Stonehenge turns out to contain 63% as much ^{14}C as a sample of present-day charcoal of equal mass. What is the age of the sample?

Solution:

We take $t = 0$ as the time of the death of the tree from which the Stonehenge charcoal was made and N_0 as the number of ^{14}C atoms that the Stonehenge sample contained then. We are given that $N = (0.63)N_0$ now, so we solve the equation $(0.63)N_0 = N_0 e^{-kt}$ with the value $k = 0.0001216$. Thus we find that

$$t = -\frac{\ln(0.63)}{0.0001216} \approx 3800 \text{ (years)}.$$

Thus the sample is about 3800 years old. If it has any connection with the builders of Stonehenge, our computations suggest that this observatory, monument, or temple—whichever it may be—dates from 1800 B.C. or earlier.