Algebra

1. If one zero of the quadratic polynomial $x^2 + 3x + k$ is 2, then the value of k is

(a) 10 (b) -10 (c) 5 (d) -5

Answer: b

2. A quadratic polynomial, whose zeroes are -3 and 4, is

(a) $x^2 - x + 12$ (b) $x^2 + x + 12$

(c) $\frac{x^2}{2} - \frac{x}{2} - 6$ (d) $2x^2 + 2x - 24$

Answer: c

3. x-a is a factor of $p(x) = ax^2+bx+c$. Which of the following is true?

(a)p(a) =2 (b)p(a)=0

(c)p(a)=5 (d)p(a)=2

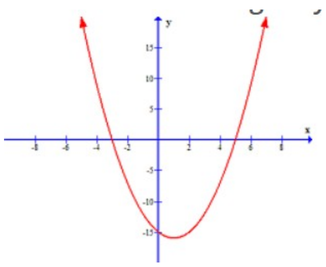
Answer: b

4. The factored form of a polynomial function is $f(x) = (x + 4)(x - 2)(x - 1)(x + 1)$. According to the Fundamental Theorem of Algebra, what is the degree of this function?

(a) 4 (b) 1 (c)7 (d) none of these

Answer: a

5. The following graph is of a polynomial function of degree 2. Are the solutions of this function real or imaginary and why?



a. The solutions are real, because the graph crosses the *x*-axis at two points.
b. The solutions are real, because the graph does not cross the *x*-axis.
c. The solutions are imaginary, because the graph crosses the *x*-axis at two points.
d. The solutions are imaginary, because the graph does not cross the *x*-axis.

Answer: a

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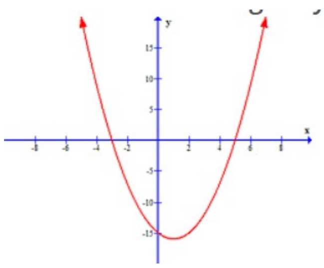

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d. The solutions are imaginary, because the graph does not cross the *x*-axis.

Answer: a

1. If $2 \cos \alpha = x + (1/x)$ and $2 \cos \beta = y + (1/y)$, show that

(i) $(x/y) + (y/x) = 2 \cos(\alpha - \beta)$

(ii) $xy - (1/xy) = 2i \sin(\alpha + \beta)$

(iii) $(x^m/y^n) - (y^n/x^m) = 2i \sin(m\alpha - n\beta)$

(iv) $(x^m y^n) + 1/(x^m y^n) = 2 \cos(m\alpha + n\beta)$

Solution:

$2 \cos \alpha = x + (1/x)$

$$x^2 + 1 = (2 \cos \alpha)\, x$$

$$x^2 - (2 \cos \alpha)\, x + 1 = 0$$

Solving for x, we get

$$= [-b + \sqrt{(b^2 - 4ac)}] / 2a$$

$$= [(2 \cos \alpha) + \sqrt{((2 \cos \alpha)^2 - 4(1)}] / 2(1)$$

$$= [(2 \cos \alpha) + \sqrt{-4 (1 - \cos^2\alpha)} / 2$$

$$= (2 \cos \alpha + i\, 2 \sin \alpha) / 2$$

$$x = \cos \alpha + i \sin \alpha$$

$2 \cos \beta = y + (1/y)$

$$y^2 + 1 = (2 \cos \beta)\, y$$

$$y^2 - (2 \cos \beta)\, y + 1 = 0$$

Solving for y, we get

$$= [-b + \sqrt{(b^2 - 4ac)}] / 2a$$

$$= [(2 \cos \beta) + \sqrt{((2 \cos \beta)^2 - 4(1)}] / 2(1)$$

$$= [(2 \cos \beta) + \sqrt{-4 (1 - \cos^2\beta)} / 2$$

$$= (2 \cos \beta + i\, 2 \sin \beta) / 2$$

$$y = \cos \beta + i \sin \beta$$

(i) $(x/y) + (y/x) = 2 \cos(\alpha - \beta)$

$xy^{-1} = (\cos \alpha + i \sin \alpha)(\cos(-\beta) + i \sin(-\beta))$

$$(x/y) = \cos(\alpha - \beta) + i \sin(\alpha - \beta)$$

$$(y/x) = \cos(\alpha - \beta) - i \sin(\alpha - \beta)$$

By adding, we get

$$(x/y) + (y/x) = 2 \cos(\alpha - \beta)$$

(ii) $xy - (1/xy) = 2i \sin(\alpha + \beta)$

$xy = (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)$

$$= \cos(\alpha + \beta) + i \sin(\alpha + \beta)$$

$1/xy = = \cos(\alpha + \beta) - i \sin(\alpha + \beta)$

$xy - (1/xy)$

$$= [\cos(\alpha + \beta) + i \sin(\alpha + \beta)] - [\cos(\alpha + \beta) - i \sin(\alpha + \beta)]$$

$$= -2i \sin(\alpha + \beta)$$

(iii) $(x^m/y^n) - (y^n/x^m) = 2i \sin(m\alpha - n\beta)$

$x = \cos \alpha + i \sin \alpha$

$$x^m = (\cos \alpha + i \sin \alpha)^m$$

$$x^m = (\cos m\alpha + i \sin m\alpha)$$

$y = \cos \beta + i \sin \beta$

$$y^n = (\cos \beta + i \sin \beta)^n$$

$$y^n = (\cos n\beta + i \sin n\beta)$$

$$(x^m/y^n) = \cos(m\alpha - n\beta) + i \sin(m\alpha - n\beta)$$

$$(y^n/x^m) = \cos(m\alpha - n\beta) - i \sin(m\alpha - n\beta)$$

$$(x^m/y^n) - (y^n/x^m) = -2i \sin(m\alpha - n\beta)$$

(iv) $(x^m y^n) + 1/(x^m y^n) = 2\cos(m\alpha + n\beta)$

$$x^m = (\cos m\alpha + i \sin m\alpha)$$

$$y^n = (\cos n\beta + i \sin n\beta)$$

$$x^m y^n = \cos(m\alpha + n\beta) + i \sin(m\alpha + n\beta)$$

$$1/(x^m y^n) = \cos(m\alpha + n\beta) - i \sin(m\alpha + n\beta)$$

$(x^m y^n) + 1/(x^m y^n) = 2\cos(m\alpha + n\beta)$

2.Solve the equation $z^3 + 27 = 0$.

Solution:

$^3 + 27 = 0$

$$z^3 = -27$$

$$z^3 = (-1 \cdot 3)^3$$

$$z = [(-1 \cdot (3)^3]^{1/3}$$

$$= 3\,(-1)^{1/3}$$

Polar form of -1 :

$-1 = 3[\cos \pi + i \sin \pi]$

$$= [\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)]$$

$$= [\cos \pi(2k + 1)) + i \sin \pi(2k + 1)]$$

$(-1)^{1/3} = [\cos \pi(2k + 1)) + i \sin \pi(2k + 1)]^{1/3}$

$$(-1)^{1/3} = [\cos(\pi/3)(2k + 1)) + i \sin(\pi/3)(2k + 1)]$$

k = 0, 1, 2

If k = 0

$$= [\cos(\pi/3)(2k + 1)) + i \sin(\pi/3)(2k + 1)]$$

$$= 3 \text{ cis } (\pi/3)$$

If k = 1

$$= 3\ [\cos \pi + i \sin \pi]$$

$$= -3$$

If k = 2

$$= [\cos (5\pi/3) + i \sin (5\pi/3)]$$

$$= 3 \text{ cis } (5\pi/3)$$

3. If $\omega \neq 1$ is a cube root of unity, show that the roots of the equation $(z-1)^3 + 8 = 0$ are -1, $1-2\omega$, $1-2\omega^2$.

Solution:

$$(z-1)^3 + 8 = 0$$

$$(z-1)^3 = -8$$

$$(z-1) = (-8)^{1/3}$$

$$(z-1) = -2 \cdot (1)^{1/3}$$

$$z = -2 \cdot (1)^{1/3} + 1$$

$$z = 1 - 2 \cdot (1)^{1/3}$$

Cube root of 1 are 1, ω, ω^2

$z = 1 - 2 \cdot 1$

$$z = 1 - 2 = -1$$

$z = 1 - 2 \cdot \omega$

$$z = 1 - 2\omega$$

$z = 1 - 2 \cdot \omega^2$

$$z = 1 - 2\omega^2$$

4. Find the value of $\sum_{k=1}^{8}(Cos\frac{2k\pi}{9}+iSin\frac{2k\pi}{9})$

Solution:

If k = 1,

$= \cos 2\pi/9 + i \sin 2\pi/9$----- (1)

If k = 2,

$= \cos 4\pi/9 + i \sin 4\pi/9$ -----(2)

If k = 3,

$= \cos 6\pi/9 + i \sin 6\pi/9$ -----(3)

If k = 4,

$= \cos 8\pi/9 + i \sin 8\pi/9$ -----(4)

....

By adding all these, we get

$= \text{cis}\ (\pi/9)\ (2 + 4 + 6 + 8 + 10 + 12 + 14 + 16)$

$= \text{cis}\ (\pi/9)\ 2(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8)$

$= \text{cis}\ (72\pi/9)$

$= \text{cis}\ 8\pi$

$= \cos 8\pi + i \sin 8\pi$

$= 1 + i(0)$

$= 1$

5. If $\omega \neq 1$ is a cube root of unity, show that

(i) $(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6 = 128$.

(ii) $(1 - \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$............$(1 + \omega^{2^{11}}) = 1$

Solution:

L.H.S:

$$= (1 + \omega^2 - \omega)^6 + (1 + \omega - \omega^2)^6$$

$$= (-\omega - \omega)^6 + (-\omega^2 - \omega^2)^6$$

$$= (-2\omega)^6 + (-2\omega^2)^6$$

$$= 64\omega^6 + 64\omega^{12}$$

$$= 64(\omega^3)^2 + 64(\omega^3)^4$$

$$= 64 + 64$$

$$= 128$$

R.H.S

Hence proved.

(ii) $(1 - \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)\ldots\ldots\ldots\ldots(1 + \omega^{2^{11}}) = 1$

L.H.S

$(1 - \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)(1 + \omega^{16})(1 + \omega^{32})(1 + \omega^{64})$

$(1 + \omega^{128})(1 + \omega^{256})(1 + \omega^{512})(1 + \omega^{1024})(1 + \omega^{2048})$

First 2 terms are $= (1 - \omega)(1 + \omega^2)$

3rd and 4th terms :

$$(1 + \omega^4)(1 + \omega^8) = (1 + \omega)(1 + \omega^2)$$

5th and 6th terms :

$$(1 + \omega^{16})(1 + \omega^{32}) = (1 + \omega)(1 + \omega^2)$$

Similarly by grouping these terms, we get

$$= [(1 + \omega)(1 + \omega^2)]^6$$

$$= [1 + \omega^2 + \omega + \omega^3]^6$$

$$= [0 + \omega^3]^6$$

$$= 1$$

Hence proved.

6. Find the inverse of matrix A given by

$$A=\begin{matrix} 1 & 1 \\ 2 & 4 \end{matrix}$$

if it exists.

Write the augmented matrix $[A|I)$

$$\begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 2 & 4 & | & 0 & 1 \end{bmatrix}$$

step 1

$$R_2 - 2 \times R_1 \begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 0 & 2 & | & -2 & 1 \end{bmatrix}$$

step 2

$$(1/2)R_2 \begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 0 & 1 & | & -1 & 1/2 \end{bmatrix}$$

step 3

$$R_1 - R_2 \begin{bmatrix} 1 & 0 & | & 2 & -1/2 \\ 0 & 1 & | & -1 & 1/2 \end{bmatrix}$$

The inverse of A is the 2 × 2 matrix on the right side given by

$$A^{-1} = \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix}$$

7. Find all eigenvalues and eigenvectors of matrix

$$A=[\begin{matrix} -2 & 1 \\ 12 & -3 \end{matrix}]$$

Find

We substitute *A, h* aod in the matrix as follows

Solve the equation

Calculate the determinant and substitute in fhe above equation

Expand and rewrite as

$$\lambda^2 + 5\lambda - 6 = 0$$

Solve the above quadratic equation to find two eigenvalues

$\lambda = 1$ and $\lambda = -6$

Find E

for - 1

Subsfifute 1 by 1 in the matrix equation *(A*- 1/)A O

$$\begin{pmatrix} -2 & 1 \\ 12 & -3 \end{pmatrix} - i \begin{matrix} 1 & C \\ 0 & . \end{matrix} = 0$$

Simplify the above

$$\begin{matrix} & 1 \\ 12 & -4 \end{matrix} \quad I{-}0$$

Let A - ** and rewrite the above matrix equation as

$$\begin{matrix} -3 & 1 & z \\ 12 & -4 & z2 \end{matrix}$$

Multiply the top equation by 4 and add it to the second equation and rewrite the system of equations

as follows

$$\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} '1 \\ 2 \end{pmatrix} - 0$$

A solution for =2 could be written as =2' where t takes all real numbers.

Use the top equation —3z —l- zt 0

to findz as follows

, 1 3

substitute 2 by t to obtain

$$x_1 = \frac{1}{3}t$$

Hence the eigenvector X corresponding the eigenvalue A — 1 may be written as

K=t [3], tE

Eigenvectors for 6

Substitute 1 by 6 in the matrix equation (A 1/)A — 0

$$\left(\begin{pmatrix} -2 & 1 \\ 12 & -3 \end{pmatrix} (6) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) W - 0$$

which may be simplified to

$$\begin{pmatrix} 4 & 1 \\ 12 & 3 \end{pmatrix} X = 0$$

Subtract 3 times the top row from the second row to obtain

$$\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix} A - 0$$

A solution for 2 could be written as 2' I where I takes all real numbers.

Use the top equation 4zt + zt 0

to find i as follows

$$x_1 = -\frac{}{4}$$

substitute z by I to obtain

$x_1 = -\frac{1}{4}t$

Hence the eigenvector X corresponding the eigenvalue $\lambda = -6$ may be written as

$$X = t\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

8. Let A be the following 3×3 matrix.

$$A=\begin{matrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & a \end{matrix}$$

We use the fact that a matrix is nonsingular if and only if it is row equivalent to the identity matrix.

We apply the elementary row operations as follows.

$$A \xrightarrow{R_3-R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & a+1 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & a+1 \end{bmatrix}.$$

If $a+1=0$, then the last matrix is in reduced row echelon form.
Thus A is not row equivalent to the identity matrix.

On the other hand, if $a+1\neq 0$, then we can continue the reduction as follows.

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & a+1 \end{bmatrix} \xrightarrow{\frac{1}{a+1}R_3} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2-2R_3]{R_1+3R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore A is row equivalent to the identity matrix.

We conclude that the matrix A is nonsingular for any values of a except for $a=-1$.

9. Express the vector $\mathbf{b}=\begin{matrix} 2 \\ 13 \\ 6 \end{matrix}$ as a linear combination of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

Solution:

We need to find numbers x_1, x_2, x_3 satisfying

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}.$$

This vector equation is equivalent to the following matrix equation.

$$[\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3]\mathbf{x} = \mathbf{b}$$

or more explicitly we can write it as

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix}.$$

Thus the problem is to find the solution of this matrix equation.
Let us consider the augmented matrix for this system to apply Gauss-Jordan elimination.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 5 & 2 & 4 & 13 \\ -1 & 1 & 3 & 6 \end{array}\right].$$

We apply elementary row operations and obtain a matrix in reduced row echelon form as follows.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 5 & 2 & 4 & 13 \\ -1 & 1 & 3 & 6 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-5R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & 3 \\ 0 & 2 & 4 & 8 \end{array}\right]$$

$$\xrightarrow[-R_2]{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & 1 & -3 \\ 0 & 1 & 2 & 4 \end{array}\right] \xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 1 & -3 \end{array}\right]$$

$$\xrightarrow[R_3-3R_2]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -5 & -15 \end{array}\right] \xrightarrow{\frac{-1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2-2R_3]{R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Therefore the solution for the system is

$$x_1=1, x_2=-2, \text{ and } x_3=3$$

and we obtain the linear combination

$$\mathbf{b} = \mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3.$$

10. Find the largest possible number of independent vectors among:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Solution: Since $\mathbf{v}_4 = \mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{v}_5 = \mathbf{v}_3 - \mathbf{v}_1$, and $\mathbf{v}_6 = \mathbf{v}_3 - \mathbf{v}_2$, the vectors $\mathbf{v}_4, \mathbf{v}_5$, and $\mathbf{v}_6$ are dependent on the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. To determine the relationship between the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ we apply row reduction to the matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

As there are three pivots, the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are independent. Therefore the largest number of independent vectors among the given six vectors is **three.** This will be the rank of the 4 by 6 matrix of $\mathbf{v}$'s.

1. If 2 cos α = x + (1/x) and 2 cos β = y + (1/y), show that

(i) $(x/y) + (y/x) = 2\cos(\alpha - \beta)$

(ii) $xy - (1/xy) = 2i\sin(\alpha + \beta)$

(iii) $(x^m/y^n) - (y^n/x^m) = 2i\sin(m\alpha - n\beta)$

(iv) $(x^m y^n) + 1/(x^m y^n) = 2\cos(m\alpha + n\beta)$

Solution:

$2\cos\alpha = x + (1/x)$

$$x^2 + 1 = (2\cos\alpha)\,x$$

$$x^2 - (2\cos\alpha)\,x + 1 = 0$$

Solving for x, we get

$$= [-b + \sqrt{(b^2 - 4ac)}] / 2a$$

$$= [(2 \cos \alpha) + \sqrt{((2 \cos \alpha)^2 - 4(1)}] / 2(1)$$

$$= [(2 \cos \alpha) + \sqrt{-4 (1 - \cos^2\alpha)} / 2$$

$$= (2 \cos \alpha + i\, 2 \sin \alpha) / 2$$

$$x = \cos \alpha + i \sin \alpha$$

$2 \cos \beta = y + (1/y)$

$$y^2 + 1 = (2 \cos \beta)\, y$$

$$y^2 - (2 \cos \beta)\, y + 1 = 0$$

Solving for y, we get

$$= [-b + \sqrt{(b^2 - 4ac)}] / 2a$$

$$= [(2 \cos \beta) + \sqrt{((2 \cos \beta)^2 - 4(1)}] / 2(1)$$

$$= [(2 \cos \beta) + \sqrt{-4 (1 - \cos^2\beta)} / 2$$

$$= (2 \cos \beta + i\, 2 \sin \beta) / 2$$

$$y = \cos \beta + i \sin \beta$$

(i) $(x/y) + (y/x) = 2 \cos (\alpha - \beta)$

$xy^{-1} = (\cos \alpha + i \sin \alpha)(\cos (-\beta) + i \sin (-\beta))$

$$(x/y) = \cos (\alpha - \beta) + i \sin (\alpha - \beta)$$

$$(y/x) = \cos (\alpha - \beta) - i \sin (\alpha - \beta)$$

By adding, we get

$$(x/y) + (y/x) = 2 \cos (\alpha - \beta)$$

(ii) $xy - (1/xy) = 2i \sin (\alpha + \beta)$

$xy = (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)$

$$= \cos (\alpha + \beta) + i \sin (\alpha + \beta)$$

$1/xy = = \cos(\alpha + \beta) - i \sin(\alpha + \beta)$

$xy - (1/xy)$

$$= [\cos(\alpha + \beta) + i \sin(\alpha + \beta)] - [\cos(\alpha + \beta) - i \sin(\alpha + \beta)]$$

$$= -2i \sin(\alpha + \beta)$$

(iii) $(x^m/y^n) - (y^n/x^m) = 2i \sin(m\alpha - n\beta)$

$x = \cos\alpha + i \sin\alpha$

$$x^m = (\cos\alpha + i \sin\alpha)^m$$

$$x^m = (\cos m\alpha + i \sin m\alpha)$$

$y = \cos\beta + i \sin\beta$

$$y^n = (\cos\beta + i \sin\beta)^n$$

$$y^n = (\cos n\beta + i \sin n\beta)$$

$$(x^m/y^n) = \cos(m\alpha - n\beta) + i \sin(m\alpha - n\beta)$$

$$(y^n/x^m) = \cos(m\alpha - n\beta) - i \sin(m\alpha - n\beta)$$

$$(x^m/y^n) - (y^n/x^m) = -2i \sin(m\alpha - n\beta)$$

(iv) $(x^m y^n) + 1/(x^m y^n) = 2\cos(m\alpha + n\beta)$

$$x^m = (\cos m\alpha + i \sin m\alpha)$$

$$y^n = (\cos n\beta + i \sin n\beta)$$

$$x^m y^n = \cos(m\alpha + n\beta) + i \sin(m\alpha + n\beta)$$

$$1/(x^m y^n) = \cos(m\alpha + n\beta) - i \sin(m\alpha + n\beta)$$

$(x^m y^n) + 1/(x^m y^n) = 2\cos(m\alpha + n\beta)$

2.Solve the equation $z^3 + 27 = 0$.

Solution:

$^3 + 27 = 0$

$$z^3 = -27$$

$$z^3 = (-1 \cdot 3)^3$$

$$z = [(-1 \cdot (3)^3]^{1/3}$$

$$= 3\,(-1)^{1/3}$$

Polar form of -1 :

$-1 = 3[\cos \pi + i \sin \pi]$

$$= [\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)]$$

$$= [\cos \pi(2k + 1)) + i \sin \pi(2k + 1)]$$

$(-1)^{1/3} = [\cos \pi(2k + 1)) + i \sin \pi(2k + 1)]^{1/3}$

$$(-1)^{1/3} = [\cos (\pi/3)(2k + 1)) + i \sin (\pi/3)(2k + 1)]$$

k = 0, 1, 2

If k = 0

$$= [\cos (\pi/3)(2k + 1)) + i \sin (\pi/3)(2k + 1)]$$

$$= 3 \text{ cis } (\pi/3)$$

If k = 1

$$= 3\,[\cos \pi + i \sin \pi]$$

$$= -3$$

If k = 2

$$= [\cos (5\pi/3) + i \sin (5\pi/3)]$$

$$= 3 \text{ cis } (5\pi/3)$$

3. If $\omega \neq 1$ is a cube root of unity, show that the roots of the equation $(z - 1)^3 + 8 = 0$ are -1, $1 - 2\omega$, $1 - 2\omega^2$.

Solution:

$$(z-1)^3 + 8 = 0$$

$$(z-1)^3 = -8$$

$$(z-1) = (-8)^{1/3}$$

$$(z-1) = -2 \cdot (1)^{1/3}$$

$$z = -2 \cdot (1)^{1/3} + 1$$

$$z = 1 - 2 \cdot (1)^{1/3}$$

Cube root of 1 are 1, ω, ω^2

$z = 1 - 2 \cdot 1$

$$z = 1 - 2 = -1$$

$z = 1 - 2 \cdot \omega$

$$z = 1 - 2\omega$$

$z = 1 - 2 \cdot \omega^2$

$$z = 1 - 2\omega^2$$

4. Find the value of $\sum_{k=1}^{8} (Cos \frac{2k\pi}{9} + iSin \frac{2k\pi}{9})$

Solution:

If k = 1,

$$= \cos 2\pi/9 + i \sin 2\pi/9 \text{----- (1)}$$

If k = 2,

$$= \cos 4\pi/9 + i \sin 4\pi/9 \text{----- (2)}$$

If k = 3,

$$= \cos 6\pi/9 + i \sin 6\pi/9 \text{ -----}(3)$$

If k = 4,

$$= \cos 8\pi/9 + i \sin 8\pi/9 \text{ -----}(4)$$

………………

By adding all these, we get

$$= cis\ (\pi/9)\ (2 + 4 + 6 + 8 + 10 + 12 + 14 + 16)$$

$$= cis\ (\pi/9)\ 2(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8)$$

$$= cis\ (72\pi/9)$$

$$= cis\ 8\pi$$

$$= \cos 8\pi + i \sin 8\pi$$

$$= 1 + i(0)$$

$$= 1$$

5. If $\omega \neq 1$ is a cube root of unity, show that

(i) $(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6 = 128$.

(ii) $(1 - \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)\text{…………}(1 + \omega^{2^{11}}) = 1$

Solution:

L.H.S:

$$= (1 + \omega^2 - \omega)^6 + (1 + \omega - \omega^2)^6$$

$$= (-\omega - \omega)^6 + (-\omega^2 - \omega^2)^6$$

$$= (-2\omega)^6 + (-2\omega^2)^6$$

$$= 64\omega^6 + 64\omega^{12}$$

$$= 64(\omega^3)^2 + 64(\omega^3)^4$$

$$= 64 + 64$$

$$= 128$$

R.H.S

Hence proved.

(ii) $(1 - \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$.............$(1 + \omega^{2^{11}}) = 1$

L.H.S

$(1 - \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)(1 + \omega^{16})(1 + \omega^{32})(1 + \omega^{64})$

$(1 + \omega^{128})(1 + \omega^{256})(1 + \omega^{512})(1 + \omega^{1024})(1 + \omega^{2048})$

First 2 terms are $= (1 - \omega)(1 + \omega^2)$

3rd and 4th terms :

$$(1 + \omega^4)(1 + \omega^8) = (1 + \omega)(1 + \omega^2)$$

5th and 6th terms :

$$(1 + \omega^{16})(1 + \omega^{32}) = (1 + \omega)(1 + \omega^2)$$

Similarly by grouping these terms, we get

$$= [(1 + \omega)(1 + \omega^2)]^6$$

$$= [1 + \omega^2 + \omega + \omega^3]^6$$

$$= [0 + \omega^3]^6$$

$$= 1$$

Hence proved.

6. Find the inverse of matrix A given by

$$A = \begin{matrix} 1 & 1 \\ 2 & 4 \end{matrix}$$

if it exists.

Write the augmented matrix $[A|I)$

$$\begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 2 & 4 & | & 0 & 1 \end{bmatrix}$$

step 1

$$R_2 - 2 \times R_1 \begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 0 & 2 & | & -2 & 1 \end{bmatrix}$$

step 2

$$(1/2)R_2 \begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 0 & 1 & | & -1 & 1/2 \end{bmatrix}$$

step 3

$$R_1 - R_2 \begin{bmatrix} 1 & 0 & | & 2 & -1/2 \\ 0 & 1 & | & -1 & 1/2 \end{bmatrix}$$

The inverse of A is the 2 × 2 matrix on the right side given by

$$A^{-1} = \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix}$$

7. Find all eigenvalues and eigenvectors of matrix

$$A=\begin{bmatrix} -2 & 1 \\ 12 & -3 \end{bmatrix}$$

Find Eigenvalues

We substitute A, λ and I in the matrix as follows

Solve the equation

Calculate the determinant and substitute in the above equation

Expand and rewrite as

$\lambda^2 + 5\lambda - 6 = 0$

Solve the above quadratic equation to find two eigenvalues

$\lambda = 1$ and $\lambda = -6$

Find E ________

Eigenvectors for - 1

Subsfifute 1 by 1 in the matrix equation (A- 1/)A O

$$\left(\begin{matrix} -2 & 1 \\ 12 & -3 \end{matrix} - i \begin{matrix} 1 & C \\ 0 & . \end{matrix} \right) - 0$$

Simplify the above

1 I–0

12

Let A - ** and rewrite the above matrix equation as

—3 1 z

12 z2

Multiply the top equation by 4 and add it to the second equation and rewrite the system of equations

as follows

—3 1

0 0 >2 = 0

A solution for zg could be written as zy — where t takes all real numbers.

Use the top equation —3 1 + 2 0

to find zt as follows

$$x_1 = \frac{}{3}$$

substitute :r2 by t to obtain

$$x_1 \quad \frac{1}{3}$$

Hence the eigenvector X corresponding the eigenvalue X 1 may be written as

$$X = t \left[\begin{matrix} 3 \end{matrix} \right], t \in \mathbb{R}$$

Eigenvectors for $\lambda = -6$

Substitute λ by 6 in the matrix equation $(A - \lambda I)X = 0$

$$\left(\begin{bmatrix} -2 & 1 \\ 12 & -3 \end{bmatrix} - (-6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) X = 0$$

which may be simplified to

$$\left(\begin{bmatrix} 4 & 1 \\ 12 & 3 \end{bmatrix}\right) X = 0$$

Subtract 3 times the top row from the second row to obtain

$$\left(\begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix}\right) X = 0$$

A solution for x_2 could be written as $x_2 = t$ where t takes all real numbers.

Use the top equation $4x_1 + x_2 = 0$

to find x_1 as follows

$$x_1 = -\frac{x_2}{4}$$

substitute x_2 by t to obtain

$$x_1 = -\frac{1}{4}t$$

Hence the eigenvector X corresponding the eigenvalue $\lambda = -6$ may be written as

$$X = t\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

8. Let A be the following 3×3 matrix.

$$A = \begin{matrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & a \end{matrix}$$

We use the fact that a matrix is nonsingular if and only if it is row equivalent to the identity matrix.

We apply the elementary row operations as follows.

$$A \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & a+1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & a+1 \end{bmatrix}.$$

If $a+1=0$, then the last matrix is in reduced row echelon form.
Thus A is not row equivalent to the identity matrix.

On the other hand, if $a+1 \neq 0$, then we can continue the reduction as follows.

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & a+1 \end{bmatrix} \xrightarrow{\frac{1}{a+1} R_3} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2-2R_3]{R_1+3R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore A is row equivalent to the identity matrix.

We conclude that the matrix A is nonsingular for any values of a except for $a=-1$.

9. Express the vector $\mathbf{b} = \begin{matrix} 2 \\ 13 \\ 6 \end{matrix}$ as a linear combination of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

Solution:

We need to find numbers x_1, x_2, x_3 satisfying

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}.$$

This vector equation is equivalent to the following matrix equation.

$$[\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3]\mathbf{x} = \mathbf{b}$$

or more explicitly we can write it as

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix}.$$

Thus the problem is to find the solution of this matrix equation.
Let us consider the augmented matrix for this system to apply Gauss-Jordan elimination.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 5 & 2 & 4 & 13 \\ -1 & 1 & 3 & 6 \end{array}\right].$$

We apply elementary row operations and obtain a matrix in reduced row echelon form as follows.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 5 & 2 & 4 & 13 \\ -1 & 1 & 3 & 6 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-5R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & 3 \\ 0 & 2 & 4 & 8 \end{array}\right]$$

$$\xrightarrow[-R_2]{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & 1 & -3 \\ 0 & 1 & 2 & 4 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 1 & -3 \end{array}\right]$$

$$\xrightarrow[R_3-3R_2]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -5 & -15 \end{array}\right] \xrightarrow{\frac{-1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2-2R_3]{R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Therefore the solution for the system is

$$x_1=1, x_2=-2, \text{ and } x_3=3$$

and we obtain the linear combination

$$\mathbf{b}=\mathbf{v}_1-2\mathbf{v}_2+3\mathbf{v}_3.$$

10. Find the largest possible number of independent vectors among:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Solution: Since $\mathbf{v}_4 = \mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{v}_5 = \mathbf{v}_3 - \mathbf{v}_1$, and $\mathbf{v}_6 = \mathbf{v}_3 - \mathbf{v}_2$, the vectors $\mathbf{v}_4, \mathbf{v}_5$, and $\mathbf{v}_6$ are dependent on the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. To determine the relationship between the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ we apply row reduction to the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

As there are three pivots, the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are independent. Therefore the largest number of independent vectors among the given six vectors is **three.** This will be the rank of the 4 by 6 matrix of $\mathbf{v}$'s.

Question6:

Show that the relation R in R defined as R = {(a, b): a ≤ b}, is reflexive and transitive but not symmetric.

Answer

R = {(a, b); a ≤ b} Clearly (a, a) ∈ R as a = a. ∴R is reflexive.

Now, (2, 4) ∈ R (as 2 < 4) But, (4, 2) ∉ R as 4 is greater than 2. ∴ R is not symmetric.

Now, let (a, b), (b, c) ∈ R. Then, a ≤ b and b ≤ c ⇒ a ≤ c ⇒ (a, c) ∈ R ∴R is transitive.

Hence, R is reflexive and transitive but not symmetric.

Question 7:

Let assume that F is a relation on the set R real numbers defined by xFy if and only if x-y is an integer. Prove that F is an equivalence relation on R.

Solution:

Reflexive: Consider x belongs to R, then x – x = 0 which is an integer. Therefore x F x.

Symmetric: Consider x and y belongs to R and x F y. Then x – y is an integer. Thus, y – x = – (x – y), y – x is also an integer. Therefore y F x.

Transitive: Consider x and y belongs to R, xFy and yFz. Therefore x-y and y-z are integers.

According to the transitive property, (x – y) + (y – z) = x – z is also an integer. So that xFz.

Thus, R is an equivalence relation on R.

Question 8:

Show that the relation R is an equivalence relation in the set A = { 1, 2, 3, 4, 5 } given by the relation R = { (a, b):|a-b| is even }.

Solution :

R = { (a, b):|a-b| is even }. Where a, b belongs to A

Reflexive Property :

From the given relation,

|a – a| = | 0 |=0

And 0 is always even.

Thus, |a-a| is even

Therefore, (a, a) belongs to R

Hence R is Reflexive

Symmetric Property :

From the given relation,

|a – b| = |b – a|

We know that |a – b| = |-(b – a)|= |b – a|

Hence |a – b| is even,

Then |b – a| is also even.

Therefore, if (a, b) ∈ R, then (b, a) belongs to R

Hence R is symmetric.

Transitive Property :

If |a-b| is even, then (a-b) is even.

Similarly, if |b-c| is even, then (b-c) is also even.

Sum of even number is also even

So, we can write it as a-b+ b-c is even

Then, a – c is also even

So,

|a – b| and |b – c| is even , then |a-c| is even.

Therefore, if (a, b) ∈ R and (b, c) ∈ R, then (a, c) also belongs to R

Hence R is transitive.

Question 9:

Show that the relation R in the set R of real numbers, defined as R = {(a, b): a ≤ b 2} is neither reflexive nor symmetric nor transitive.

Answer

R = {(a, b): a ≤ b 2} It can be observed that ∴R is not reflexive.

Now, (1, 4) ∈ R as 1 < 42 But, 4 is not less than 12 . ∴(4, 1) ∉ R ∴R is not symmetric.

Now, (3, 2), (2, 1.5) ∈ R (as 3 < 22 = 4 and 2 < (1.5)2 = 2.25) But, 3 > (1.5)2 = 2.25 ∴(3, 1.5) ∉ R ∴ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 10:

Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as R = {(a, b): b = a + 1} is reflexive, symmetric or transitive.

Answer

Let A = {1, 2, 3, 4, 5, 6}.

A relation R is defined on set A as: R = {(a, b): b = a + 1} ∴R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)} We can find (a, a) ∉ R, where a ∈ A. For instance, (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) ∉ R ∴R is not reflexive.

It can be observed that (1, 2) ∈ R, but (2, 1) ∉ R. ∴R is not symmetric.

Now, (1, 2), (2, 3) ∈ R But, (1, 3) ∉ R

∴R is not transitive

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question11

Find the eigenvalues and eigenfunctions for the differential equation y 00 + λy = 0 with the following boudary conditions

y(0) = 0, y 0 (1) = 0

Answer

y 00 + λy = 0 (∗) There are three different cases to consider; λ < 0, λ = 0 and λ > 0.

We consider them each in turn.

λ < 0. In this case we let λ = −k 2 , (where k 6= 0) and equation (*) becomes y 00 − k 2 y = 0, ⇒ y = A cosh(kx) + B sinh(kx).

Using the boundary condition y(0) = 0,⇒ A = 0, so y = B sinh(kx).

Differentiating gives y 0 = kB cosh(kx) and hence y 0 (1) = 0 gives kB cosh k = 0 ⇒ B = 0 (since k 6= 0).

So there is no non-trivial solution if λ < 0.

(ii) λ = 0. In this case the equation becomes y 00 = 0,⇒ y = A + Bx. The boundary condition y(0) = 0 ⇒ A = 0, and the boundary condition y 0 (1) = 0,⇒ B = 0.

So there is no non-trivial solution for λ = 0. (iii) λ > 0.

In this case we let λ = k 2 (where k 6= 0) and equation (*) becomes y 00 + k 2 y = 0, ⇒ y = A cos(kx) + B sin(kx). 1 The boundary condition y(0) = 0 gives A = 0, so that y = B sin(kx).

Hence y 0 = Bk cos(kx) and the boundary condition y 0 (1) = 0 gives Bk cos k = 0.

For non-trivial solutions we require cos k = 0 and hence k = (2n − 1)π 2 , n = 1, 2, 3, ...The corresponding eigenvalues and eigenfunctions are λn = (2n − 1)2π 2 4 , yn = sin " (2n − 1)πx 2 # , n = 1, 2, 3, ...

Question12

Find the eigenvalues and eigenfunctions for the differential equation y 00 + λy = 0 with the following boudary conditions

y 0 (0) = 0, y(1) = 0

ANS

As in part (a) there are no non-trivial solutions unless λ > 0.

We write λ = k 2 (with k 6= 0) and (*) becomes y 00 + k 2 y = 0, with solution y = A cos(kx) + B sin(kx), and derivative y 0 = −Ak sin(kx) + Bk cos(kx). Using the boundary condition y 0 (0) = 0 gives B = 0 and hence y = A cos(kx). Using the boundary condition y(1) = 0 gives A cos k = 0 so for nontrivial solutions we require: cos k = 0, ⇒ k = (2n − 1)π 2 , n = 1, 2, 3, . . .

The corresponding eigenvalues and eigenfunctions are λn = (2n − 1)2π 2 4 , yn = sin " (2n − 1)πx 2 # , n = 1, 2, 3, ...

Question14

Find the eigenvalues and eigenfunctions for the differential equation y 00 + λy = 0 with the following boudary conditions

y 0 (0) = 0, y 0 (1) = 0

ANS

There are no non-trivial solutions when λ < 0, however in this case there are non-trivial solution when λ = 0 or λ > 0.

When λ = 0 the equation becomes y 00 = 0 with solution y = A + Bx. Hence y 0 = B and the boundary condition y 0 (0) = 0 requires B = 0

. However with this condition the other boundary condition y 0 (1) = 0 is automatically satisfied and hence y = A satisfies the DE and the boundary conditions.

Hence λ0 = 0 is an eigenvalue with y0 = 1 the corresponding eigenfunction.

For λ > 0 the solution is as usual y = A cos(kx) + B sin(kx), with derivative y 0 = −Ak sin(kx) + Bk cos(kx). The boundary condition y 0 (0) = 0 gives B = 0 and hence 2 y 0 = −Ak sin(kx) The other boundary condition y 0 (1) = 0 now gives −Ak sin k = 0 so for non-trivial solutions we require: sin k = 0, ⇒ $k = n\pi$, n = 1, 2, 3, ...

The corresponding eigenvalues and eigenfunctions are $\lambda_n = n^2\pi^2$, $y_n = \cos(n\pi)$ n = 1, 2, 3, ...Note that if we allow n = 0 this includes the case of the zero eigenvalue.

Question15

Find the eigenvalues and eigenfunctions for the differential equation $y'' + \lambda y = 0$ with the following boudary conditions

$y'(0) = 0$, $y'(1) + y(1) = 0$

ANS

As in part (a) there are no non-trivial solutions unless $\lambda > 0$. We write $\lambda = k^2$ (with $k \neq 0$) and (*) becomes $y'' + k^2 y = 0$, with solution $y = A\cos(kx) + B\sin(kx)$, and derivative $y' = -Ak\sin(kx) + Bk\cos(kx)$

. The boundary condition $y'(0) = 0$ gives B = 0 so $y = A\cos(kx)$ and $y' = -Ak\sin(kx)$.

Applying the second boundary condition $y(1) + y'(1) = 0$ gives $\cos k - k\sin k = 0$ which implies $k = \cot k$. By drawing the graphs of $y = \cot x$ and $y = x$ we see that $k = \cot k$ has an infinite number of positive roots; $k_1, k_2, k_3, \ldots$.

The corresponding eigen values and eigen functions are $\lambda_n = k_n^2$, $y_n = \cos(k_n\pi)$ n = 1, 2, 3, ... Where k_n is the n-th positive root of $x = \cot x$

. Note this eigenvalue problem arises in a problem in quantum mechanics.