Q 1. $\varphi \in$ Aut G is called an inner automorphism if there is a $g \in G$ such that $\varphi = \varphi_g$. Two elements $a, b \in G$ are called conjugate if there is a $g \in G$ such that $\varphi_g(a) = b$. We write $a \sim b$. Show that $\sim$ is an equivalence relation.

Q2 Show that for $a, b \in G$ the elements ab and ba are conjugate.

Q3Determine all automorphisms of the Klein four group.

Q4Exercise. Show that the symmetry group of a rectangle (that is not a square) is the Klein four group

Q5. Exercise. Set $\zeta := e\ 2\pi i/n$, and let $G = \{\zeta\ k : k = 1, \ldots, n\}$ be the group of the nth roots of unity, where the composition is standard multiplication in C.

(1) Show that $\varphi : Z \rightarrow G$, m 7→ ζ m, is a homomorphism.

(2) Calculate ker(φ).

Q6 Let G be a group. Show that: (1) If Aut G = {id} then G is abelian.

Q 7 If x 7→ x 2 defines an homomorphism of G, then G is abelian.

Q 8 If x 7→ x −1 defines an automorphism of G, then G is abelian

Q9 G=H×K be a nonabelian group that $o(H)=p_2$, $o(K)=p_3$ (p is prime number), why is $o(G')=p$?

Q10 If G/Z(G) is cyclic, then G is abelian.
If |G| is a p-power (p prime), then $Z(G)\neq 1$

Q11 What is an example of an automorphism of a group G that does not belong to Inn(G), the group of all inner automorphisms?

Q12 Prove that inner automorphism of Abelian groups is identity. $f{:}g\rightarrow g_{-1}$ is a automorphism for Abelian groups.

Q13 If a cyclic group has an element of infinite order, how many elements of finite order does it have?

Q14 Let G=<g> be a cyclic group of infinite order, where $g\in G$ is a generator. What can you say about the order of the element $g_k \in G$? Consider the cases k=0 and $k\neq 0$.

Q15 Prove that Every cyclic group is abelian.

Q16 If g is an element of infinite order in a group, then g k = g ` if and only if k = `. If g is an element of finite order m in a group, then g k = g ` if and only if k ≡ ` mod m.

Q17 An infinite cyclic group has exactly two generators

Q18 Let G be a cyclic group with only one generator. Then G has at most two elements. To see this, note that if g is a generator for G, then so is g −1 . If G has only one generator, it must be the case that g = g −1 . But then g 2 = e. Since g generates G, it follows that G has at most two elements.

Q19 Let H be a nonempty finite subset of a group G. Prove that if H is closed, then it is a subgroup. To see this, suppose $h \in H$. Since H is closed, h n $\in$ H for every positive integer n, and since H is finite, there exist n, m with n > m such that $h^n = h^m$. Then $h^{n-m} = e$, so $e \in$ H. Furthermore, $h^{-1} = h^{n-m-1} \in H$. Thus H is a subgroup.

Q20 Suppose a and b are elements of a group G and that ab has order n. Then ba also has order n. To see this, note that $(ba)^n = a^{-1}(ab)^n a$. Thus if $(ab)^n = e$, then $(ba)^n = e$. The converse holds by symmetry, and so ab and ba have the same order.

Q21 Suppose G is a group which has only finitely many subgroups. We want to prove that G is finite. First of all, recall that if $g \in G$, then $< g >$ is a cyclic group containing g. If $< g >$ is infinite, then $< g >$ is isomorphic to Z. But the group Z has infinitely many subgroups (one for each natural number), and then G would also have infinitely many subgroups, a contradiction. Hence each $< g >$ is finite. Since G has only finitely many subgroups, and since each of these is finite, and since every element of G is contained in a finite group, G has only finitely many elements.

Q22 The group z4 has the property that every proper subgroup is cylic, but it itself is not cyclic.

Q23 Suppose that G is a group and $a \in G$ is the unique element of order 2. Then ax = xa for all $x \in G$. To see this, let $b := xax^{-1}$. Then $a = x^{-1}bx$. Since a 6= e, b 6= a, and an easy calculation shows that $b^2 = e$, so it has order 2. Since a is unique, a = b, and this implies that ax = xa

Q24 For a prime p, any element of GL2(Z/(p)) with order p is conjugate to a strictly upper-triangular matrix (1 a 0 1). The number of p-Sylow subgroups is p + 1.

Q25 The number of elements of order p in GL2(Z/(p)) is p ^2 -1

Q26 If #G = 20 or 100 then G has a normal 5-Sylow subgroup

Q27 . Prove G = SL2(Z/(3)). This group has size 24 and a normal 2-Sylow subgroup

Q28 A group of size 70 has a normal subgroup of size 35

Q29 Let G1 and G2 be groups of order 24 and 30, respectively. Let G3 be a nonabelian group that is a homomorphic image of both G1 and G2. Describe G3, up to isomorphism.

Q30 Prove that a finite group whose only automorphism is the identity map must have order at most two.

Q31 Let H be a nontrivial subgroup of Sn. Show that either $H \subseteq An$, or exactly half of the permutations in H are odd.

Q32 Let p be a prime number, and let A be a finite abelian group in which every element has order p. Show that Aut(A) is isomorphic to a group of matrices over Zp.

Q33 Let G be a group and let N be a normal subgroup of G of finite index. Suppose that H is a finite subgroup of G and that the order of H is relatively prime to the index of N in G. Prove that H is contained in N