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[This question paper contains 4 printed pages]

Your Roll No. : ...2019.....

Sl. No. of Q. Paper : 7464 J

Unique Paper Code : 32351302

Name of the Course : **B.Sc.(Hons.)
Mathematics**

Name of the Paper : Group Theory - I

Semester : III

Time : 3 Hours **Maximum Marks : 75**

Instructions for Candidates :

- (a) Write your Roll No. on the top immediately on receipt of this question paper.
- (b) Attempt any **two** parts from each question.
- (c) All questions carry equal marks.

1. (a) Let $G = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in \mathbb{R}, a \neq 0 \right\}$. Show that G is

a group under matrix multiplication.

(b) Let G be a group and H be a subset of G . Prove that H is a subgroup of G if $a, b \in H \Rightarrow ab^{-1} \in H$. Hence prove that $H = \{A \in G : \det A \text{ is a power of } 3\}$ is a subgroup of $GL(2, \mathbb{R})$.

P.T.O.

(c) (i) Suppose G is a group that has exactly eight elements of order 3. How many subgroups of order 3 does G have?

(ii) If $|a| = n$ and k divides n , prove that $|a^{n/k}| = k$.

$$6 \times 2 = 12$$

2. (a) Let $G = \langle a \rangle$ be a cyclic group of order n . Prove that $G = \langle a^k \rangle$ if and only if $\gcd(k, n) = 1$. List all the generators of Z_{20} .

(b) (i) If a cyclic group has an element of infinite order, how many elements of finite order does it have.

(ii) List all the elements of order 6 and 8 in Z_{30} .

(c) Suppose that a and b are group elements that commute and have orders m and n . If $\langle a \rangle \cap \langle b \rangle = \{e\}$, Prove that the group contains an element whose order is the least common multiple of m and n . Show that this need not be true if a and b do not commute.

$$6.5 \times 2 = 13$$

3. (a) Let G be a group. Is $H = \{x^2 : x \in G\}$ a subgroup of G ? Justify.

(b) Prove that any two left cosets of a subgroup H in a group G are either equal or disjoint.

(c) Show that $(\mathbb{Q}, +)$ has no proper subgroup of finite index.

$$6 \times 2 = 12$$

4. (a) Prove that every subgroup of index 2 is normal. Show that A_5 is normal subgroup of S_5 .
- (b) Let G be a group and H be a normal subgroup of G . Prove that the set of all left cosets of H in G forms a group under the operation $aH \cdot bH = abH$ where $a, b \in G$.
- (c) If H is a normal subgroup of G with $|H| = 2$, prove that $H \subseteq Z(G)$. Hence or otherwise show that A_5 cannot have a normal subgroup of order 2. 6.5 × 2 = 13
5. (a) Let C be the set of complex numbers and

$$M = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Prove that C and M are isomorphic under addition and $C^* = C \setminus \{0\}$ and $M^* = M \setminus \{0\}$ are isomorphic under multiplication.

- (b) Prove that a finite cyclic group of order n is isomorphic to the group $Z_n = \{0, 1, 2, \dots, n-1\}$ under addition modulo n .
- (c) (i) Suppose that φ is an isomorphism from a group G onto a group G^* . Prove that G is cyclic if and only if G^* is cyclic.

(ii) Show that Z , the group of integers under addition is not isomorphic to Q , the group of rationals under addition. $6 \times 2 = 12$

6. (a) Let φ be a group homomorphism from a group G to a group G^* then prove that :

(i) $|\varphi(x)|$ divides $|x|$, for all x in G .

(ii) φ is one-one if and only if $|\varphi(x)| = |x|$, for all x in G .

(b) State and prove the Third Isomorphism Theorem.

(c) (i) Let G be a group. Prove that the mapping $\phi(g) = g^{-1}$, for all $g \in G$, is an isomorphism on G if and only if G is Abelian.

(ii) Determine all homomorphisms from Z_n to itself.

$$6.5 \times 2 = 13$$