RING THEORY AND LINEAR ALGEBRA

Q:1:-Let(V,+v,.v,0v) be a K-vector space. Then, the following properties are satisfied for all vϵV and all aϵK;

(a) 0.v u=0.v;
(b) a.v 0v=0v;
(c) a.v u=0v;
(d) (-1) .vu=-v

Proof. (a) $0 \cdot_V v = (0+0) \cdot_V v = 0 \cdot_V v + 0 \cdot_V v$, donc $0 \cdot_V v = 0_V$.
(b) $a \cdot_V 0_V = a \cdot_V (0_V + 0_V) = a \cdot_V 0_V + a \cdot_V 0_V$, donc $a \cdot_V 0_V = 0_V$.
(c) Assume $a \cdot_V v = 0_V$. If $a = 0$, the assertion $a = 0 \vee v = 0_V$ is true. Assume therefore $a \neq 0$. Then a^{-1} has a meaning. Consequently, $v = 1 \cdot_V v = (a^{-1} \cdot a) \cdot_V v = a^{-1} \cdot_V (a \cdot_V v) = a^{-1} \cdot_V 0_V = 0_V$ by (b).
(d) $v +_V (-1) \cdot_V v = 1 \cdot_V v +_V (-1) \cdot_V v = (1 + (-1)) \cdot_V v = 0 \cdot_V v = 0_V$ by (a). □

Q:2:-Let V be a K-vector space and E is subset of V a non emptysubspace,we set

$$\langle E \rangle := \{\sum_{i=1}^{m} a_i e_i \mid m \in \mathbb{N}, e_1, \ldots, e_m \in E, a_1, \ldots, a_m \in K\}.$$

This sis a vector subspace of V, said to be generated by E.

Proof. Since $\langle E \rangle$ is non-empty (since E is non-empty), it suffizes to check the definition of subspace. Let therefore $w_1, w_2 \in \langle E \rangle$ and $a \in K$. We can write

$$w_1 = \sum_{i=1}^{m} a_i e_i \text{ et } w_2 = \sum_{i=1}^{m} b_i e_i$$

for $a_i, b_i \in K$ and $e_i \in E$ for all $i = 1, \ldots, m$. Thus we have

$$a \cdot w_1 + w_2 = \sum_{i=1}^{m} (aa_i + b_i) e_i,$$

which is indeed an element of $\langle E \rangle$. □

Q:3:-**How to compute the intersection of two subspace?**

(a) *The easiest case is when the two subspaces are given as the solutions of two systems of linear equations, for example:*

- V *is the subset of* $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in K^n$ *such that* $\sum_{i=1}^{n} a_{i,j} x_i = 0$ *for* $j = 1, \dots, \ell$, *et*
- W *is the subset of* $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in K^n$ *such that* $\sum_{i=1}^{n} b_{i,k} x_i = 0$ *pour* $k = 1, \dots, m$.

In this case, the subspace $V \cap W$ *is given as the set of common solutions for all the equalities.*

(b) *Suppose now that the subspaces are given as subspaces of* K^n *generated by finite sets of vectors: Let* $V = \langle E \rangle$ *and* $W = \langle F \rangle$ *où*

$$E = \left\{ \begin{pmatrix} e_{1,1} \\ e_{2,1} \\ \vdots \\ e_{n,1} \end{pmatrix}, \dots, \begin{pmatrix} e_{1,m} \\ e_{2,m} \\ \vdots \\ e_{n,m} \end{pmatrix} \right\} \subseteq K^n \text{ and } F = \left\{ \begin{pmatrix} f_{1,1} \\ f_{2,1} \\ \vdots \\ f_{n,1} \end{pmatrix}, \dots, \begin{pmatrix} f_{1,p} \\ f_{2,p} \\ \vdots \\ f_{n,p} \end{pmatrix} \right\} \subseteq K^n.$$

Then

$$V \cap W = \{ \sum_{i=1}^{m} a_i \begin{pmatrix} e_{1,i} \\ e_{2,i} \\ \vdots \\ e_{n,i} \end{pmatrix} \mid$$

$$\exists b_1, \dots, b_p \in K : a_1 \begin{pmatrix} e_{1,1} \\ e_{2,1} \\ \vdots \\ e_{n,1} \end{pmatrix} + \dots + a_m \begin{pmatrix} e_{1,m} \\ e_{2,m} \\ \vdots \\ e_{n,m} \end{pmatrix} - b_1 \begin{pmatrix} f_{1,1} \\ f_{2,1} \\ \vdots \\ f_{n,1} \end{pmatrix} - \dots - b_p \begin{pmatrix} f_{1,p} \\ f_{2,p} \\ \vdots \\ f_{n,p} \end{pmatrix} = 0 \}.$$

Here is a concrete example: $E = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subseteq K^n$ *and* $F = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\} \subseteq K^n$. *We have to solve the system*

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = 0.$$

With operations on the rows, we obtain

$$\ker(\begin{pmatrix} 1 & 0 & -1 & -2 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & -1 \end{pmatrix}) = \ker(\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}) = \ker(\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}),$$

thus we obtain as solution subspace the line generated by $\begin{pmatrix} -1 \\ 1 \\ -3 \\ 1 \end{pmatrix}$, *so the intersection is given by the line*

$$\langle -1 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle = \langle -3 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \rangle = \langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \rangle.$$

Here is the alternative characterization of the subspace generated by a set

Q:4:-Let V be a K-vector space and E is subspace of V a non- empty subset. Then we have the equality

$$\langle E \rangle = \bigcap_{W \leq V \text{ subspace s.t. } E \subseteq W} W$$

Where the right hand side is the intersection of all subspaces W of V containing E.

Proof. To prove the equality of two sets, we have to prove the two inclusions.
' $\subseteq$ ': Any subspace W containing E, also contains all the linear combinations of elements of E, hence W contains $\langle E \rangle$. Consequently, $\langle E \rangle$ in the intersection on the right.
' $\supseteq$ ': Since $\langle E \rangle$ belongs to the subspaces in the intersection on the right, it is clear that this intersection is contained in $\langle E \rangle$. □

Q:5:- Let V be a K-vector space, Wi≤V subspace of V for iϵI≠ϕ and W=∑iϵIWi.Then the following assertions are equivalent

(i) $W = \bigoplus_{i \in I} W_i$ *;*

(ii) for all $w \in W$ *and all* $i \in I$ *there exists a unique* $w_i \in W_i$ *such that* $w = \sum'_{i \in I} w_i$.

Proof. "(i) $\Rightarrow$ (ii)": The existence of such $w_i \in W_i$ is clear. Let us thus show the uniqueness

$$w = \sum_{i \in I}{}' w_i = \sum_{i \in I}{}' w_i'$$

with $w_i, w_i' \in W_i$ for all $i \in I$ (remember that the notation $\sum'$ indicates that only finitely many w_i, w_i' are non-zero). This implies for $i \in I$:

$$w_i - w_i' = \sum_{j \in I \setminus \{i\}}{}' (w_j' - w_j) \in W_i \cap \sum_{j \in I \setminus \{i\}} W_j = 0.$$

Thus, $w_i - w_i' = 0$, so $w_i = w_i'$ for all $i \in I$, showing uniqueness.
"(ii) $\Rightarrow$ (i)": Let $i \in I$ and $w_i \in W_i \cap \sum_{j \in I \setminus \{i\}} W_j$. Then, $w_i = \sum'_{j \in I \setminus \{i\}} w_j$ with $w_j \in W_j$ for all $j \in I$. We can now write 0 in two ways

$$0 = \sum_{i \in I}{}' 0 = -w_i + \sum_{j \in I \setminus \{i\}}{}' w_j.$$

Hence, the uniqueness imples $-w_i = 0$. Therefore, we have shown $W_i \cap \sum_{j \in I \setminus \{i\}} W_j = 0$. $\square$

Q:6:-How to compute a basis for a vector space generated by a finite set of vectors? Solve a system of linear equations.

- *Add e_1 to the basis.*

- *If e_2 is linearly independent from e_1 (i.e. e_2 is not a scalar multiple of e_1), add e_2 to the basis and in this case e_1, e_2 are linearly independent (otherwise, do nothing).*

- *If e_3 is linearly independent from the vectors chosen for the basis, add e_3 to the basis and in this case the elements chosen for the basis are linarly independent (otherwise, do nothing).*

- *If e_4 is linearly independent from the vectors already chosen for the basis, add e_4 to the basis and in this case all the chosen elements for the basis are linearly independent (otherwise, do nothing).*

- *etc. until the last vector.*

Here is a concrete example in $\mathbb{R}^4$:

$$e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, e_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 2 \end{pmatrix}, e_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

- *Add e_1 to the basis.*

- *Add e_2 to the basis since e_2 is clearly not a multiple of e_1 (see, for example, the second coefficient), thus e_1 et e_2 are linearly independent.*

- *Are e_1, e_2, e_3 linearly independent?We consider the system of linear equations given by the matrix*

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix}.$$

By transformations on the rows, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain the solution $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. So, we do not add e_3 to the basis since e_3 is linearly dependent from e_1, e_2.

- *Are e_1, e_2, e_4 linearly independent? We consider the system of linear equations given by the matrix*

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

By transformations on the rows, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The corresponding system has no non-zero solution. Therfore e_1, e_2, e_4 are linearly independent. This is the basis that we looked for.

Q:7:-let nϵN

Let $M = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$ *be a matrix with n columns, m rows and with coefficients in K (we denote the set of these matrices by $\mathrm{Mat}_{m\times n}(K)$; this is also a K-vector space). It defines the K-linear map*

$$\varphi_M : K^n \to K^m, \quad v \mapsto Mv$$

where Mv is the usual product for matrices. Explicitely,

$$\varphi_M(v) = Mv = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n a_{1,i}v_i \\ \sum_{i=1}^n a_{2,i}v_i \\ \vdots \\ \sum_{i=1}^n a_{m,i}v_i \end{pmatrix}.$$

The K-linearity reads as

$$\forall a \in K \, \forall v, w \in V : M \circ (a \cdot v + w) = a \cdot (M \circ v) + M \circ w.$$

Q:8:-Write augmented matrix.

M=

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 4 & 5 & 6 \\ 0 & 0 & 1 & 7 & 8 & 9 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 3 \\ -4 & 1 & 0 & 0 & -3 & -6 \\ -7 & 0 & 1 & 0 & -6 & -12 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 3 \\ -4 & 1 & 0 & 0 & -3 & -6 \\ 1 & -2 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 3 \\ 4/3 & -1/3 & 0 & 0 & 1 & 2 \\ 1 & -2 & 1 & 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} -5/3 & 2/3 & 0 & 1 & 0 & -1 \\ 4/3 & -1/3 & 0 & 0 & 1 & 2 \\ 1 & -2 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The left half of the final matrix is the matrix C looked for: $C = \begin{pmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 1 & -2 & 1 \end{pmatrix}$. *The right half is the matrix obtained by the operations on the rows.*

We know that we have the following equality (to convince ourselves, we can verify it by a small computation):

$$CM = \begin{pmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

As application of the Gauß's algorithm written in terms of matrices, we obtain that any invertible square matrix M can be written as product of the matrices of Definition 1.39. Indeed, that we can transform M into identity by operations on the rows.

Q:9:-Find the eigenvalues

M=((2,1,1,),(3,2,3),(-3,-2,-1))

For the eigenvalue 1, *we compute the kernel*

$$\ker\left(\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ -3 & -1 & -2 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \ker\left(\begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 3 \\ -3 & -1 & -3 \end{pmatrix}\right)$$
$$= \ker\left(\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \ker\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \langle\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\rangle$$

For the eigenvalue -1, *we compute the kernel*

$$\ker\left(\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ -3 & -1 & -2 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \ker\left(\begin{pmatrix} 3 & 1 & 1 \\ 3 & 3 & 3 \\ -3 & -1 & -1 \end{pmatrix}\right)$$
$$= \ker\left(\begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\right) = \ker\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right) = \langle\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\rangle$$

For the eigenvalue 2, *we compute the kernel*

$$\ker\left(\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ -3 & -1 & -2 \end{pmatrix} - 2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \ker\left(\begin{pmatrix} 0 & 1 & 1 \\ 3 & 0 & 3 \\ -3 & -1 & -4 \end{pmatrix}\right) = \ker\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right) = \langle\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\rangle$$

Q:10:-Let φ∈EndK(V). The following statements are equivalent:

(i) φ is diagonalizable.

(ii) V = L λ∈Spec(φ) Eφ(λ).

Proof. "(i) ⇒ (ii)": We have the inclusion $\sum_{\lambda\in\mathrm{Spec}(\varphi)} E_\varphi(\lambda) \subseteq V$. By Lemma 3.10, the sum is direct, therefore we have the inclusion $\bigoplus_{\lambda\in\mathrm{Spec}(\varphi)} E_\varphi(\lambda) \subseteq V$. Since φ is diagonalizable, there exists a K-basis of V consisting of eigenvectors for φ. Thus, any element of this basis already belongs to $\bigoplus_{\lambda\in\mathrm{Spec}(\varphi)} E_\varphi(\lambda)$, whence the equality $\bigoplus_{\lambda\in\mathrm{Spec}(\varphi)} E_\varphi(\lambda) = V$.
"(ii) ⇒ (i)": For all $\lambda \in \mathrm{Spec}(\varphi)$ let S_λ be a K-basis of the eigenspace $E_\varphi(\lambda)$. Thus $S = \bigcup_{\lambda\in\mathrm{Spec}(\varphi)} S_\lambda$ is a K-basis of V consisting of eigenvectors, showing that φ is diagonalizable. □