COMPLEX ANALYSIS

Q:1:-Show that if we calculate with symbols x+iy, where x and y are real numbers, according to the usual rules for adding and multiplying numbers and in addition use i 2 = −1, then all the requirements for a field are satisfied.

From now on the field we have constructed is denoted by C and called the field of complex numbers. Note that the field of real numbers is an ordered field. This means that we have a relation < defined among the real numbers such that (1) If x and y ∈ R, then exactly one of x < y, y < x and x = y is true. (2) Sums and products of positive (i.e., > 0) numbers are positive. We have not introduced anything similar for the complex numbers for the simple reason that it can not be done.

Q:2:-Write 1+2i /3+4i on standard form.

The geometric interpretation of addition is already familiar, since this is the ordinary vector addition in the plane. To get a geometric picture of multiplication, we introduce polar coordinates in the plane in the following way. If z 6= 0, then z/|z| is located somewhere on the unit circle; hence we can find an angle θ such that z/|z| = cosθ+isin θ. We may therefore write z on polar form as z = |z|(cos θ + isin θ) where θ is called the argument of z and is denoted θ = arg z. It is unfortunatebut extremely important that arg z is NOT uniquely determined by z; adding any integer multiple of 2π to θ gives another, equally valid, value for arg z. When one therefore speaks of 'the' argument for a complex number, one means one of the infinitely many possible values of the argument. Another, less serious ambiguity, is that we have not assigned an argument to the number 0; it is usual to allow any real number whatsoever as a valid argument for 0.

Now suppose z = |z|(cos θ + isin θ) and w = |w|(cos ϕ + isin ϕ) are complex numbers. Then zw = |z||w|(cos θ cos ϕ − sin θ sin ϕ + i(cos θ sin ϕ + sin θ cos ϕ)) = |zw|(cos(θ + ϕ) + isin(θ + ϕ)) according to the addition formulas for sin and cos. Thus, when calculating the product of two complex numbers the absolute values are multiplied and the arguments are added. In particular, multiplication by a complex number of absolute value 1 is equivalent to a rotation with an angle equal to the argument of the given number.

Q:3:-The image of a straight line in C under stereographic projection is a circle through N, with N excluded. The image of a circle in C under stereographic projection is a circle not containing N. The inverse image of any circle on S2 is a straight line together with ∞ if the circle passes through N, otherwise a circle.

Proof. Since a straight line in the x1x2-plane together with N determines a unique plane, the intersection of which with S2 is the image of the straight line we only need to consider the case of a circle in C. If it has center a and radius r its equation is |z − a| 2 = r ^{2} or |z| 2 −2 Re(az) +|a| ^{2} = r ^{2} . Substituting z = x1+ix2 1−x3 into this, using that x 2 1 + x 2 2 + x 2 3 = 1 and x3 6= 1, we get 1 + x3 − 2x1 Re a − 2x2 Im a + (1 − x3)(|a| ^{2} − r ^{2}) = 0 which is the equation of a plane. Conversely, a circle on the Riemann sphere is determined by three distinct points. The inverse images of these three points determine a circle in C. The image of this circle is clearly the original circle.

Q:4:-Stereographic projection is conformal.

Proof. Consider two curves intersecting at z and their tangents at z in C. Together with N the tangents determine two planes that intersect the Riemann sphere in two circles through N. The tangents to the circles at N are in these planes and also in the plane through N parallel to C. It follows that they are parallel to the original tangent vectors so that viewed from inside the sphere they give rise to an angle equal to but of opposite orientation to the original angle.

The circles intersect also at the image of z on the sphere, and are tangent to the images of the curves there. The angles at the two points where the circles intersect are equal but of opposite orientation by symmetry (the two angles are images of each other under reflection in the plane through the origin and parallel to the normals of the planes of the circles).

Q:5:-Suppose f is continuous in a region Ω. Then f has a primitive F in Ω if and only if Z γ f(z) dz = 0 for every closed arc γ ⊂Ω. It is enough if this is true for arcs made up solely of vertical and horizontal line segments.

Proof. If F is a primitive of f in Ω and γ a closed arc with initial and final points z1 = z0, then R γ f(z) dz = F(z1) − F(z0) = 0 since z1 = z0. Conversely, if the integral along closed arcs vanishes, pick a point z0 ∈Ω and define F(z) = R γ f, where γ is an arc in Ω starting at z0 and ending at z. This gives an unambiguous definition of F, since if γ˜ is another such arc, then the arc γ − γ˜ is a closed arc in Ω. Thus the integral along γ has the same value as the integral along γ˜. We may restrict ourselves to arcs of the special type of the statement of the theorem, since in an open, connected set Ω every pair of points may be connected by an arc of this kind in Ω (show this as an exercise!). It now remains to show that F is a primitive of f in Ω. 40 3. INTEGRATION Writing z = x + iy with real x, y we shall calculate the partial derivatives of F with respect to x and y. To do this, let h ∈ R be so small that the line segment between z and z + h is contained in Ω. Then F(z +h)−F(z) = R h 0 f(z +t) dt. This is seen by choosing an arc γ starting at z0 and ending at z to calculate F(z), and then calculating F(z + h) by adding to γ the line segment connecting z to z + h, which we parametrize by z(t) = z + t, 0 ≤ t ≤ h. By the fundamental theorem of calculus, differentiating with respect to h gives F 0 x (z) = f(z). Similarly, considering F(z + ih) − F(z) = i R h 0 f(z + it) dt we obtain F 0 y (z) = if(z). Thus the Cauchy-Riemann equation F 0 x + iF0 y = 0 is satisfied, and F 0 = F 0 x = f, so that F is a primitive of f.

Q:6:-Suppose f is analytic in a closed rectangle R except for at an interior point p, where (z − p)f(z) → 0 as z → p. If γ is the positively oriented boundary of R, then Z γ f(z) dz = 0.

Proof. Let ε > 0 and R0 ⊂ R be a square centered at p and so small that |(z − p)f(z)| < ε for z ∈ R0. If γ0 is the positively oriented boundary of R0 we obtain ¯ ¯ ¯ Z γ0 f(z) dz ¯ ¯ ¯ ≤ ε Z γ0 |dz| |z − p| ≤ 8ε. The last inequality is due to the facts that |z − p| ≥ \`/2 if \` is the side length of R0, and that the length of γ0 is 4\`. Now extend the sides of R0 until they cut R into 9 rectangles, one of which is R0. The other 8 satisfy the assumptions of Theorem 3.4. It follows that | R γ f| = | R γ0 f| ≤ 8ε, and since ε > 0 is arbitrary the integral over γ must be 0.

Q:7:- Zeros of an analytic function not identically 0 are isolated points in the domain of analyticity.

Proof. Suppose f is analytic at p and f(p) = 0. According to Theorem we may expand f in power series P∞ k=0 ak(z −p) k . Since f(p) = 0 the first term in the series vanishes, and if n is the first index for which an 6= 0 we obtain f(z) = (z −p) n g(z), where g is the analytic function P∞ k=0 an+k(z−p) k , so that g(p) = an 6= 0. The positive integer n is called to order or multiplicity of the zero p. Since g is continuous and g(p) 6= 0 there is a neighborhood of p in which g doesn't vanish. Since (z −p) n only vanishes for z = p it follows that there is a neighborhood of p in which p is the only zero of f.

Q:8:- Suppose f is analytic and has no zeros in a simply connected region Ω. Then one may define a branch of log(f(z)) in Ω.

Proof. Since f has no zeros in Ω the function f 0 (z)/f(z) is analytic in Ω so that Cauchy's integral theorem applies to it. According to Theorem 3.3 there is therefore a primitive g of this function defined in Ω, and d dz (f(z)e −g(z)) = f 0 (z)e −g(z) − f(z) f 0 (z) f(z) e −g(z) = 0, so that f(z)e −g(z) = C, where C 6= 0 since neither f nor the exponential function vanishes. Thus we may find A ∈ C so that e A = C. It follows that f(z) = e g(z)+A, so that g(z) + A is a branch of log(f(z)).

Q:9:-The range of the restriction of an analytic function to an arbitrary punctured neighborhood of an essential singularity is dense in C.

Proof. Suppose f is analytic in the punctured neighborhood Ω of a, and that there is a complex number b such that all values of f in Ω has distance at least d > 0 from b. Consider the function g(z) = (f(z) − b) −1 . It is analytic in Ω and bounded by 1/d there. By Theorem 4.1 it therefore has a removable singularity at a so that 1/g(z) has at most a pole at a (if g has a zero of order n at a, then the pole has order n). So, f(z) = b + 1/g(z) has at worst a pole at a.

Q:10:-We wish to find the number of zeros in the first quadrant of the polynomial f(z) = $z^4 - z^3 +$ $13z^2 - z + 36$.

Solution: First note that there are no zeros on either the real or imaginary axes since for z = x ∈ R we have $x^4 - x^3 + 13x^2 - x + 36 = (x^2 + 1)(x - 1\ 2\ )^2$ + 47 /4 x 2 + 143 /4 > 0 and for z = iy, y ∈ R we have $z^4 - z^3 + 13z^2 - z + 36 = y^4 - 13y^2 + 36 + i(y^3 - y)$. The imaginary part vanishes only for y = 0 and y = ±1, neither of which is a zero for the real part. Now let γ be the line segment from 0 to R > 0, followed by a quarter circle of radius R centered at 0 and ending at iR, and finally the vertical line segment from iR to 0. For R sufficiently large, all the zeros in the first quadrant will be inside γ, so we only need to calculate the variation of argument for the polynomial along γ. Since f > 0 on the real axis, the argument stays equal to 0 along the horizontal part of γ. For |z| = R we write f(z) = z^4 (1 − 1 z + 13 z^2 − 1 z^3 + 36 ^{z} 4). Note that the bracketed expression tends to 1 as R → ∞ so its argument varies only a little around 0. The argument of the first factor varies 4 times the variation of the argument of z, i.e., by 4 π 2 = 2π. So, along the circular arc the argument varies close to 2π. It remains to find the variation of the argument along the imaginary axis. If ϕ denotes the argument of f(z), then tan ϕ = y 3−y y 4−13y 2+36 . For y = 0 this is 0, and for y → ∞ we get tan ϕ → 0. The argument variation along the vertical part of γ is therefore close to some integer multiple of π. To go from one multiple to the next, tan ϕ will have to become ∞ in between. This happens at the zeros of $y^4 - 13y^2 + 36 = (y^2 - 9)(y^2 - 4) = (y + 3)(y + 2)(y - 3)(y - 2)$. The first two factors stay positive for y ≥ 0 so the denominator in tan ϕ passes from positive to negative as y decreases through 3, and from negative to positive as y decreases past 2. In both these points the numerator is positive, so tan ϕ passes from +∞ to −∞ as y decreases through 3 and then from −∞ back to +∞ as y decreases through 2. Hence, if we start at y = R for a large value of R, the variation in argument along the vertical line segment is close to 0. Therefore, for large R > 0 the variation in argument of f along γ is close to 2π, and since it has to be an integer multiple of 2π, it is exactly 2π. There is therefore exactly one zero of f in the first quadrant.