[This question paper contains 7 printed pages]

Your Roll No. : ....

Sl. No. of Q. Paper : **7466** **J**

Unique Paper Code : 32351501

Name of the Course : **B.Sc.(Hons.) Mathematics**

Name of the Paper : Metric Spaces

Semester : V

Time : 3 Hours ***Maximum Marks : 75***

Instructions for Candidates :

(a) Write your Roll No. on the top immediately on receipt of this question paper.

(b) Attempt any **two** parts from each question.

1. (a) Define a metric space. Let $p \geq 1$. Define $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as $d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$, $x\ (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Show that $(\mathbb{R}^n, d_p)$ is a metric space. 6.5

P.T.O.

(b) When is a metric space said to be complete ? Is discrete metric space complete ? Justify. 6.5

(c) Let (X, d) be a metric space. Define $d_1: X \times X \to \mathbb{R}$ by $d_1(x,y)=\dfrac{d(x,y)}{1+d(x,y)}$, for all $x, y \in X$.

Prove that d_1 is a metric on X and d_1 is equivalent to d. 6.5

2. (a) Prove that every open ball in a metric space (X, d) is an open set in (X, d). What about the converse ? Justify. 6

(b) Define a homeomorphism from a metric space (X,d_1) to a metric space (Y, d_2). Show that the function $f:\mathbb{R} \to]-1, 1[$ defined by $f(x)=\dfrac{x}{1+|x|}$ is a homeomorphism. 6

(c) Let (X, d) be a metric space and let A, B be non-empty subsets of X. Prove that : 6

(i) $(A \cap B)^0 = A^0 \cap B^0$

(ii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

3. (a) Let (X, d) be a metric space and $F \subseteq X$. Prove that the following statements are equivalent : 6

(i) $x \in \bar{F}$

(ii) $S(x, \varepsilon) \cap F \neq \phi$, for every open ball $S(x, \varepsilon)$ centred at x

(iii) There exists an infinite sequence $\{x_n\}$ of point (not necessarily distinct) of F such that $x_n \to X$.

(b) Let (X,d) be a metric space and $F \subseteq X$. Prove that F is closed in X if and only if F^C is open in X, where F^C is complement of F in X. 6

(c) Let (X,d) be a metric space such that for every nested sequence $\{F_n\}_{n\geq 1}$ of non-empty closed subsets of X satisfying $d(F_n) \to 0$ as $n \to \infty$, the intersection $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point. Prove that (X,d) is complete. 6

4. (a) Let f be a mapping from a metric space (X, d_1) to a metric space (Y, d_2). Prove that f is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y. 6.5

(b) Let (X, d_1) and (Y, d_2) be two metric spaces. Prove that the following statements are equivalent : 6.5

(i) f is continuous on X

(ii) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, for all subsets B of Y

(iii) $f(\overline{A}) \subseteq \overline{f(A)}$, for all subsets A of X.

(c) Define uniform continuity of a function f from a metric space (X, d_1) to a metric space (Y, d_2). Let (X, d) be a metric space and A be a non-empty subset of X. Show that the function $f : (X, d) \rightarrow \mathbb{R}$ defined as $f(x) = d(x, A)$, for all $x \in X$, is uniformly continuous on X.

6.5

5. (a) State and prove contraction mapping theorem. 6

(b) (i) Let Y be a non-empty subset of a metric space (X, d) and (Y, d_y) be complete, where d_y is restriction of d to $Y \times Y$. Prove that Y is closed in X. 3

(ii) Let A be a non-empty bounded subset of a metric space (X, d). Prove that $d(A) = d(\bar{A})$. 3

P.T.O.

(c) Let (X, d) be a metric space. Then prove that following statements are equivalent :

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(i) (X, d) is disconnected.

(ii) There exist two non-empty disjoint subsets A and B, both open in X, such that $X = A \cup B$.

(iii) There exist two non-empty disjoint subsets A and B, both closed in X, such that $X = A \cup B$.

(iv) There exists a proper subset of X, which is both open and closed in X.

6\. (a) Let ($\mathbb{R}$, d) be the space of real numbers with the usual metric. Show that a connected subset of $\mathbb{R}$ must be an interval. Give example of two connected subsets of $\mathbb{R}$ such that their union is disconnected.

6.5

(b) Let (X, d) be a metric space and Y be a subset of X. If Y is compact subset of (X, d), then prove that Y is closed. 6.5

(c) Let f be a continuous function from a compact metric space (X, d_1) to a metric space (Y, d_2). Prove that f is uniformly continuous on X.

6.5