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[This question paper contains 4 printed pages]

Your Roll No. : 2019

Sl. No. of Q. Paper : **7463** **J**

Unique Paper Code : 32351301

Name of the Course : **B.Sc.(Hons.) Mathematics**

Name of the Paper : Theory of Real Functions

Semester : III

Time : 3 Hours ***Maximum Marks : 75***

Instructions for Candidates :

(a) Write your Roll No. on the top immediately on receipt of this question paper.

(b) Attempt any **three** parts from each question.

(c) **All** questions carry equal marks.

1. (a) Find the following limit and establish it by using $\epsilon-\delta$ definition of limit :

$$\lim_{x \to -1} \frac{x+5}{2x+3}$$

(b) State and prove the sequential criterion for limits of a real valued function.

(c) Determine whether the following limit exists in R :

$$\lim_{x \to 0} \operatorname{sgn}\left(\sin 1/x^2\right)$$

(d) Show that :

$$\lim_{x \to 0^-} e^{\frac{1}{x}} = 0$$

and establish that

$$\lim_{x \to 0^+} e^{\frac{1}{x}}$$

does not exist in R .

2. (a) Let $c \in R$ and f be defined on (c, ∞) and $f(x) > 0$ for all $x \in (c, \infty)$. Show that

$$\lim_{x \to \infty} f(x) = \infty$$

if and only if

$$\lim_{x \to \infty} \frac{1}{f(x)} = 0$$

(b) Evaluate the following limit by using the appropriate definition :

$$\lim_{x \to 1^+} \frac{x}{x-1}$$

(c) Determine the points of continuity of the function f (x) = x [x] where [.] denotes the greatest integer function.

(d) A function $f : R \to R$ is such that f (x+y) = f(x) + f(y) for all x,y in R. Prove that if f is continuous at some point x_0, then it is continuous at every point of R.

3. (a) Let $A \subseteq R$ and $f : A \to R$ and let $f(x) \geq 0$, for all $x \in A$. Let $\sqrt{f}$ be defined as $\sqrt{f}(x) = \sqrt{f(x)}$ for $x \in A$. Show that if f is continuous at a point $c \in A$, then $\sqrt{f}$ is continuous at c.

(b) Suppose that $f : R \to R$ is continuous on R and that f(r) = 0 for every rational number r. Prove that f(x) = 0 for all $x \in R$.

(c) Let f be a continuous and real valued function defined on a closed and bounded interval [a, b]. Prove that f is bounded. Give an example to show that the condition of boundedness of the interval cannot be dropped.

(d) State the intermediate value theorem. Show that $x_2^k = 1$ for some $x \in]0, 1[$.

4. (a) Show that the function $f(x) = x^2$ is uniformly continuous on $[-2, 2]$, but it is not uniformly continuous on R.

(b) Prove that if f and g are uniformly continuous on $A \subseteq R$ and if they both are bounded on A, then their product fg is uniformly continuous on A.

(c) Show that the function

$f(x) = |x + 1| + |x - 1|$

is not differentiable at -1 and 1.

(d) Prove that if $f : R \to R$ is an even function and has a derivative at every point, then the derivative f′ is an odd function.

5. (a) State Darboux theorem. Let I be an interval and $f : I \to R$ be differentiable on I. Show that if the derivative f' is never zero on I, then either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.

(b) Find the Taylor's series for cos x and indicate why it converges to cos x for all $x \in R$.

(c) Prove that $e^x \geq 1 + x$ for all $x \in R$, with equality occurring if and only if $x = 0.5$.

(d) Is $f(x) = |x|$, $x \in R$, a convex function ? Is every convex function differentiable ? Justify your answer.

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