GROUP THEORY II

MULTIPLE CHOICE QUESTIONS

1. If G and H are two groups , then GÅH is a abelian

a) G is abelian

b) H is abelian

c) G and H both are abelian

d) None is abelian

Ans: c)

2. If |U(8)||U(10)|=16, then what is |U(8) ÅU(10)|?

a) 8

b) 10

c) 16

d) 80

Ans: c)

3. ZmÅZn is isomorphic to Zmniff

a) gcd(m,n)=1

b) lcm(m,n)=1

c) lcm(m,n)=mn

d) m=n

Ans: a)

4. If |G| = 2p where p is prime and p>2, then G is isomorphic to

a)Zp

b) Z2p or Dp

c)Up

d)U2p

Ans: b)

5. Determine the number of elements of order 10 in Z100ÅZ25.

a)100

b)25

c)5

d)24

Ans: d)

6. The order of any non identity element in Z5ÅZ5ÅZ5 is:

a)25

b)5

c)125

d)15

Ans: b)

7. The number of automorphisms of order 12 in Aut(Z720):

a) 360

b) 120

c) 96

d) 720

Ans: c)

8. Let G ={1,7,17,23,49,55,65,71} under multiplication modulo 96 then G is isomorphic to

a) Z8

b) Z4ÅZ2

c) Z2ÅZ2ÅZ2

d) None

Ans: b)

9. If G is infinite cyclic grp then, order of Aut(G) =

A) 2

B) 0

C) 1

D) none

Ans-B

Q10. Inn(D6) is isomorphic to

A) z6

B) D3

C) z4

D) none

Ans-B

Q11. Every subgroup of cyclic group G is characteristic if

A) G is finite

B) G is infinite

C) Both 1 & 2

D)none

Ans – C

Q12. Any subgroup H of G is called a characteristic subgroup if Φ(H) =

A) H

B) e (identity)

C) G

D) None

Ans- A

Q13. Let G and H be distinct groups then, Φ : G -> H can't be

A) Isomorphism

B) Homomorphism

C) Automorphism

D) None

Ans- C

Q14. Let G be a finite cyclic group with 12 elements. Then O(Aut(G))=

A) 4

B) 11

C) 12

D) 2

Ans- A

Q15. Subgroup G generated by all commutators [u, v] such that u, v ∈ G then it is known as

A) Normal Commutator Subgroup

B) Commutator Normal Subgroup

C) Commutator subgroup

D) Normal subgroup

Ans- C

SUBJECTIVE QUESTIONS

Q1. Find an isomorphism from the group of integers under addition to the group ofeven integers under addition.

Solution -

Let 2Z be the set of all even integers. Define a map $\varphi : Z \to 2Z$ as $\varphi(n) = 2n$.

We claim that φ is an isomorphism. $\varphi(n) = \varphi(m) \Rightarrow 2n = 2m \Rightarrow n$

$= m$ so it is

one-to-one. For any even integer 2k, $\varphi(k) = 2k$ thus it is onto. Also

$\varphi(n + m) = 2(n + m) = 2n + 2m = \varphi(n) + \varphi(m)$,

so it has the operation preserving property.

Q2. Show that U(8) is not isomorphic to U(10).

Solution -

U(10) = {1, 3, 7, 9} is a cyclic group generated by 3. So 3 is an element of order

4. But all non-identity elements of U(8) = {1, 3, 5, 7} have order 2, so there is no

element of order 4. Therefore they are not isomorphic to each other.

Q3. Show that U(8) is isomorphic to U(12).

Solution -

$U(8) = \{1, 3, 5, 7\}$ and $U(12) = \{1, 5, 7, 11\}$. Take a bijective map $\varphi : U(8) \to$

$U(12)$ defined by $\varphi(1) = 1$, $\varphi(3) = 11$, $\varphi(5) = 5$, and $\varphi(7) = 7$. We claim that it has

the operation preserving property. Because U(8) is an Abelian group, it suffices

to check followings:

$\varphi(3^2) = \varphi(1) = 1 = 11^2 = \varphi(3)^2$,

$\varphi(5^2) = \varphi(1) = 1 = 5^2 = \varphi(5)^2$,

$\varphi(7^2) = \varphi(1) = 1 = 7^2 = \varphi(7)^2$,

$\varphi(3 \cdot 5) = \varphi(7) = 7 = 11 \cdot 5 = \varphi(3) \cdot \varphi(5)$,

$\varphi(3 \cdot 7) = \varphi(5) = 5 = 11 \cdot 7 = \varphi(3) \cdot \varphi(7)$,

$\varphi(5 \cdot 7) = \varphi(3) = 11 = 5 \cdot 7 = \varphi(5) \cdot \varphi(7)$.

In general, you may show that any bijective map $\psi : U(8) \to U(12)$ with $\psi(1) = 1$

is an isomorphism.

Q4. Let φ be an isomorphism from a group G to a group G and let a belong to G.

Prove that $\varphi(C(a)) = C(\varphi(a))$.

Solution-

Let $x \in \varphi(C(a))$. Then there is $y \in C(a)$ such that $\varphi(y) = x$. $x\varphi(a) = \varphi(y)\varphi(a) =$

$\varphi(ya) = \varphi(ay) = \varphi(a)\varphi(y) = \varphi(a)x$. So $x \in C(\varphi(a))$. Therefore $\varphi(C(a)) \subset$

$C(\varphi(a))$.

Conversely, suppose that $x \in C(\varphi(a))$. There is $y \in G$ such that $\varphi(y) = x$. $\varphi(ya) =$

$\varphi(y)\varphi(a) = x\varphi(a) = \varphi(a)x = \varphi(a)\varphi(y) = \varphi(ay)$. Because φ is one-to-one, $ya =$

ay and $y \in C(a)$. Therefore $x = \varphi(a) \in \varphi(C(a))$. So $C(\varphi(a)) \subset \varphi(C(a))$ and

$C(\varphi(a)) = \varphi(C(a))$.

Q5. Find Aut(Z6).

Solution –

Z6 is a cyclic group generated by 1. There are two generators, 1, 5. So for an

isomorphism $\varphi : Z6 \to Z6$, $\varphi(1) = 1$ or $\varphi(1) = 5 = -1$. We have two such

isomorphisms: the identity map and $\varphi(x) = -x$.

Therefore Aut(Z6) = {id, φ}

where φ(x) = −x.

Q6. Suppose that φ : Z20 → Z20 is an automorphism and φ(5) = 5. What are thepossibilities for φ(x)?

Solution –

Because an automorphism φ maps a generator to a generator, φ(1) is one of

1, 3, 7, 9, 11, 13, 17, 19. Because φ(5) = φ(5·1) = 5φ(1) = 5 in Z20, the only possible

φ(1) are 1, 9, 13, 17. Therefore φ(x) = x, 9x, 13x, or 17x.

Q7. Prove or disprove that U(20) and U(24) are isomorphic.

Solution –

In U(20), 3^2 = 9, 3^3 = 27 = 7, 3^4 = 81 = 1. So |3| = 4. On the other hand, in U(24),

all non-identity elements have order two. Therefore they are not isomorphic to

each other.

Q8. Prove that S4 is not isomorphic to D12.

Solution –

Note that **D12** has an element of order 12 (rotation by 30 degrees), while **S4** has **no** element of order 12. Since orders of elements are preserved under **isomorphisms ,S4**cannot be **isomorphic to D12.**

9. If G1,G2,.....,Gn are groups ,then |G1ÅG2Å......ÅGn|is ?

Ans: |G1Å**G2**Å**.....Gn|=|G1||G2|.....|Gn|.**

10. What is the order of an element in EDP of G1,G2,...Gn?

Ans: Let g1 be any element of G1,g2 be any element of G2 and so on gn be any element of Gn

Then,

|(g1,g2....,gn)|=lcm(|g1|,|g2|,....|gn|)

11. What is the condition for GÅH to be cyclic?

Ans : G+H is cyclic iff |G| and |H| are relatively prime.

12. Determine the possible isomorphism class of Z60.

Ans: 60=2^2×3×5

Z60 is isomorphic to Z4ÅZ3ÅZ5

Or Z2ÅZ2ÅZ3ÅZ5

13. Write U(105) as the EDP of more groups

Ans: U(105)=U(3×5×7)

gcd(3,5,7)=1

U(105) is isomorphic to

U(3) ÅU(5ÅU(7)

U(15) ÅU(7)

U(21) ÅU(5)

14. What is the largest order of any element in Z3ÅZ9?

Ans: Let (a,b) be any element of Z3ÅZ9

Largest possible value for a and b is 3 and 9 respectively

And lcm(3,9)=9

Largest order of (a,b)=9

15. How many abelian groups (up to isomorphism) are there of order 15?

Ans: 15=3×5

~ Z3ÅZ5

Which means only one such abelian group is possible for order 15.