RING THEORY AND LINEAR ALGEBRA-I

MULTIPLE CHOICE QUESTIONS.

1. For the group Sn of all permutations of n distinct symbols, what is the number of elements in Sn?
a) n
b) n-1
c) 2n
d) n!

2. Does the set of residue classes (mod 3) form a group with respect to modular addition?
a) Yes
b) No
c) Can’t Say
d) Insufficient Data

3. Does the set of residue classes (mod 3) form a group with respect to modular addition?
a) Yes
b) No
c) Can’t Say
d) Insufficient Data

4. For the group Sn of all permutations of n distinct symbols, Sn is an abelian group for all values of n.
a) True
b) False

5. In modular arithmetic : (a/b) = b(a^-1)
a) True
b) False

Answers.

1. d) Explanation: There there are n distinct symbols there will be n! elements.

2. a) Explanation: Yes. The identity element is 0, and the inverses of 0, 1, 2 are respectively 0, 2, 1.

3. b) Explanation: No. The identity element is 1, but 0 has no inverse

4. b) Explanation: For n>2 it does not form a Abelian Group.

5. b) Explanation: This statement is not true. The correct version would be : (a/b) = a(b^{-1}).

SUBJECTIVE QUESTIONS-

1. Give a specific example of a prime ideal in the ring Q[x] which is not a maximal ideal.
   SOLUTION- The zero ideal in Q[x] is a prime ideal because Q[x] is an integral domain. However, the zero ideal in Q[x] is not a maximal ideal because Q[x] is not a field.

2. Suppose that R is an integral domain and that a, b ∊ R. We say that a and b are "relatively prime" if (a) + (b) = R. Suppose that c ∊ R. Assume that a and b are relatively prime and that a|bc in R. Prove that a|c in R.
   SOLUTION. We will give two arguments. First of all, since (a) + (b) = R, there exist elements s, t ∊ R such that

$$sa + tb = 1_R$$

   Multiply this equation by c. We obtain c = sac + tbc. Note that sac = (sc)a is a multiple of a in R and hence is in the ideal (a). Furthermore, bc is a multiple of a in R (as stated in the problem) and hence bc is in the ideal (a). Thus, t(bc) is in (a) too. It follows that sac + tbc∊ (a). That is, c ∊ (a). Therefore, a|c in R, as we wanted to prove.

Alternatively, we can use the result in problem 4. Let I = (b) and J = (a). We have
I +J = R. Thus b+Jis a unit in the ring R/J. Since a|bc in R, we have bc∈ J. Therefore, we have

$$(b + J)(c + J) = bc + J = 0_R + J$$

in the ring R/J. However, b + J is a unit in the ring R/J. Multiplying by the inverse of
b + J, we find that c + J = 0_R + J. That is, we have c ∈ J. This means that c is a multiple of a in R. Therefore, a|c in R, as we wanted to prove.

3. 3. Suppose that R is a commutative ring with identity and that K is an ideal of R. Let R′ = R/K. The correspondence theorem gives a certain one-to-one correspondence between the set of ideals of R containing K and the set of ideals of R′. If I is an ideal of R containing K, we let I′ denote the corresponding ideal of R′. Show that if I is principal, then so is I′. Show by example that the converse is not true in general.

SOLUTION. Let ϕ: R → R′ be defined by ϕ(r) = r + K. Then ϕ is a surjective ring homomorphism from R to R′ . Suppose that I is an ideal of R which contains K. The corresponding ideal in R′ is $\phi(I) = \{ \phi(i) \mid i \in I \}$.

Suppose that I is a principal ideal in R. Then I = (a) for some $a \in R$. That is, we have $I = \{ ra \mid r \in R \}$. Then

$$I' = \phi(I) = \{\phi(ra) \mid r \in R\} = \{\phi(r)\phi(a) \mid r \in R\} = \{r'\phi(a) \mid r' \in R'\}$$

The last equality is true because $\phi : R \to R'$ is a surjective map. It follows that
$I' = \phi(a)$, the principal ideal in R′ generated by $\phi(a)$.

4. Suppose that R is an integral domain with identity. Suppose that I and J are ideals in R and that I = (b) where $b \in R$. Prove that I + J = R is and only if b + J is a unit in the ring R/J.
SOLUTION. First of all, assume that I + J = R. Then there exists $i \in I$ and $j \in J$ such that $i + j = 1_R$. Furthermore, since $i \in (b)$, we have $i = rb$ for some $r \in R$. Therefore, we have $rb + j = 1_R$. This implies that $1_R \in rb + J$. Therefore, we have

$$1_R + J = rb + J = (r + J)(b + J)$$

The multiplicative identity element in R/J is $1_R + J$. Note that since R is a commutative ring, so is R/J. It follows that

$$(r + J)(b + J) = 1_R + J \text{ and } (b + J)(r + J) = 1_R + J$$

It follows that b + J is indeed a unit in the ring R/J. Its inverse in that ring is r + J. Now assume that b + J is a unit in the ring R/J. Thus, for some $r \in R$, we have

$$(r + J)(b + J) = 1_R + J$$

Thus, $rb + J = 1_R + J$ and hence $1_R \in rb + J$. Thus, $1_R = rb + j$ for some $j \in J$.
Let i = rb. Since I = (b), it follows that $i \in I$. Thus,

$$1_R = i + j \in I + J$$

and therefore, for any $s \in R$, we have $s = s1_R \in I + J$. It follows that I + J = R, as we wanted to prove.

5. Give an explicit example of an injective ring homomorphism from Z/5Z to Z/20Z. No justification of your answer is needed.

SOLUTION. We will justify the answer. One idempotent in the ring Z/20Z is 16 + 20Z. This element is an idempotent because

$$(16 + 20Z)(16 + 20Z) = 256 + 20Z = 16 + 20Z$$

Notice also that 16 + 20Z has order 5 in the additive group of Z/20Z. We define a map $\phi : Z \to Z/20Z$ as follows:

$$\phi(n) = 16n + 20Z$$

for $n \in Z$. As discussed in class, this map ϕ is a ring homomorphism from Z/20Z. Since 16 + 20Z has order 5, we have $Ker(\phi) = 5Z$. By the first isomorphism theorem, we obtain an injective ring homomorphism $\psi : Z/5Z \to Z/20Z$ defined by

$$\psi(n + 5Z) = 16n + 20Z$$

6. Suppose that I and J are ideals in a ring R. Assume that $I \cup J$ is an ideal of R. Prove that $I \subseteq J$ or $J \subseteq I$.

SOLUTION. Assume to the contrary that I is not a subset of J and that J is not a subset of I. It follows that there exists an element $i \in I$ such that $i \notin J$. Also, there exists an element $j \in J$ such that $j \notin I$. Note that $i \in I \cup J$ and $j \in I \cup J$. Since we are assuming that $I \cup J$ is an ideal of R, it follows that $i + j \in I \cup J$.
Let $k = i + j$. If $k \in I$, then $k - i \in I$ too. That is, $j \in I$. This is not true and hence $k \notin I$. If $k \in J$, then $k - j \in J$ too. That is, $i \in J$. However, this is not true and hence $k \notin J$. We have shown that $k \notin I$ and $k \notin J$. That is, $k \notin I \cup J$. Thus, $i + j \notin I \cup J$, contradicting what was found in the previous paragraph.
This contradiction proves the stated assertion.

7. If in a ring R every $x \in R$ satisfies $x^2 = x$, prove that R must be commutative.
SOLUTION. Let $x, y \in R$. Then $(x + y)^2 = (x + y)(x + y) = x^2 + xy + yx + y^2$.
Since $x^2 = x$ and $y^2 = y$ we have $x + y = x + xy + yx + y$.
Hence $xy = -yx$.
But for every $x \in R$:

$(-x) = (-x)^2 = (-x)(-x) = x^2 = x$. Hence $-yx = yx$ i.e. we obtain $xy = yx$.

8. If R is a ring and a, b, c, d ∈ R, evaluate (a + b)(c + d).

SOLUTION. $(a + b)(c + d) = a(c + d) + b(c + d)$ by distributive law

$= (ac + ad) + (bc + bd)$

$= ac + ad + bc + bd$

9. Show that the commutative ring D is an integral domain if and only if for a, b, c ∈ D with a ≠ 0 the relation ab = ac implies that b = c.

SOLUTION. If D is a commutative ring and a ≠ 0, then ab = ac implies a(b − c) = 0. Since a ≠ 0 we obtain b = c.

Conversely assume that ab = ac and a ≠ 0 implies that b = c. Assume if possible that a ≠ 0 and ab = 0 Then ab = a0 and hence b = 0.

10. If U is an ideal of R, let r(U) = {x ∈ R | xu = 0 for all u ∈ U}. Prove that r(U) is an ideal of R.

SOLUTION. Let $x_1, x_2 \in r(U)$ and let u ∈ U. Then $(x_1 - x_2)u = x_1u - x_2u = 0$ as $x_1u = 0$ and $x_2u = 0$.

Hence $x_1 - x_2 \in r(U)$. Now let $r \in R$ and $x \in r(U)$.
Then
$(rx)u = r(xu) = 0$
$(xr)u = x(ru) = 0$ as U is an ideal, $ru \in U$.
Hence r(U) is an ideal of R.

LINEAR ALGEBRA

1) Let *V* be the vector space of all 6×6 real matrices over the field *R* .Then the dimension of the subspace of *V* consisting of all symmetric matrices is
   (a) 15 (b) 18 (c) 21 (d) 35
   Ans: (c)

   W= Collection of all symmetric matrices

   A' = Transpose of A

   W= { A belongs to V : *A' = A }*

   Therefore,

   dim*W* = n(n+1)/ 2

   = 6(6+1)/2 = 21 .

2) Consider the subspace

$W = \{(x_1, x_2, \ldots\ldots, x_{10})$ belongs to R^{10} : $x_n = x_{n-1} + x_{n-2}$ for $3 \leq n \leq 10\}$ of the vector space R^{10}. The dimension of W is

(a) 2 (b) 3 (c) 9 (d) 10

Ans: (a)

For all i belongs to {3,4,.....,10}

X_i depends on x_1 & x_2 .

Hence, dim W = 10-8 =2 .

3) Let $T : R^2 \rightarrow R^2$ be a linear transformation such that $T((1,2))=(2,3)$ and $T((0,1))=(1,4)$. Then $T((5,6))$ is

(a) (6,-1) (b) (-6,1) (c) (-1,6) (d) (1,-6)

Ans: (a)

(5,6) = a(1,2) + b(0,1)

a = 5 , 2a + b = 6

b = 6-10 = -4

$T(5,6) = 5T(1,2) - 4T(0,1)$ (T is linear)

= 5(2,3) -4(1,4)

$T(5,6) = (6,-1)$.

4) Let $T : R^3 \to R^3$ be the linear transformation defined by T(x ,y, z) = (x + y , y + z , z + x) for all (x, y, z) belongs to R^3. Then

(a) Rank (T) = 0 , nullity (T) = 3
(b) Rank (T) = 2 , nullity (T) = 1
(c) Rank (T) = 1 , nullity (T) = 2
(d) Rank (T) = 3 , nullity (T) = 0

Ans: (d)

T(x , y ,z) = (x + y , y + z , z + x)

If T(x , y , z) = (0 , 0 , 0)

x + y = 0 => x = -y

y + z = 0 => y = -z

z + x = 0 => z = -x

⇨ x = y = z = 0

n(T) = 0 =>p(T) = 3 (by rank nullity theorem) .

5) If A and B are 3×3 real matrices such that rank(AB) = 1 , then rank(BA) can't be

(a) 0 (b) 1 (c) 2 (d) 3

Ans: (d)

Rank(AB) = 1 => AB is singular =>det(AB)=0

⇨ det (A) = 0 or det (B) = 0

⇨ det (BA) = 0 => rank (*BA*) ≠ 3 .

6) Suppose Q belongs to $M_{3\times3}(R)$ is a matrix of rank 2. Let *T*: $M_{3\times3}(R)$-> $M_{3\times3}(R)$ be the linear transformation defined by *T*(P) = PQ. Then the rank of T is _________.

Solution:

Given, Q belongs to $M_{3\times3}$ *(R) and p*(Q)=2

Now, T: $M_{3\times3}$ R ->$M_{3\times3}$ *R defined by T(P)=PQ*

Then, *p*(T) = 3 × p(Q) = 3×2 = 6 .

7) Let *A* be a 3×3 real matrix with det(A) = 6 . Then find det(*adjA*).

Solution:

adj (A) = det (A)A^{-1}

⇨ det(adj (A)) = $|A|^3$det (A^{-1})

= 216× 1/6 = 36 .

8) Let P be a 7×7 matrix of rank 4 with real entries . Let a belongs to R^7 be a column vector. Then the rank of $P + aa^T$ *is at least* ________.

Solution:

p(P) = 4

If a=0 ,$p(aa^T) = 0$
If a≠0 ,$p(aa^T) = 1$
Also, $p(A+B) \geq | p(A) – p(B) |$
If a = 0 ,$p(P + aa^T) \geq 4$
If a ≠ 0 ,$p(P + aa^T) \geq 3$
Therefore, $p(P + aa^T)$ is atleast 3 .

9) Let *u* and *v* be the eigenvectors of A corresponding to the eigen values 1 and 3 respectively . Prove that *u*+*v*is not an eigen vector of A.

Solution :

As *u , v* are eigen vectors of A corresponding to eigen values 1,3 respectively

Therefore , Au = u and Av = 3v

Let if possible u+v is an eigen vector of A corresponding to eigen value k , then

- ⇨ A(u + v) = k(u + v)
- ⇨ u + 3v = k(u + v)
- ⇨ (1-k)u + (3-k)v =0

Since u , v are eigen vectors corresponding to distinct eigen values therefore they must be linearly independent => 1-k = 0 = 3-k

⇨ 1=3 which is not possible .

Therefore u + v cannot be eigen vector of A.

10) Let P and Q be two real matrices of size 4×6 and 5×4 , respectively . If rank(Q) = 4 and rank(QP) = 2 , then rank(P) is equal to _____.

Solution:

p(P) = p(QP)

If Q is non- singular matrix .

11) Let V be a real n-dimensional vector space and let T: V -> V be a linear transformation satisfying $T^2(v) = -v$ for all v belongs to V. Show that n is even .

Solution :

Given T: V->V such that $T^2(v) = -v$, for all v belongs to V

⇨ $(T^2 + I)v = 0$, for all v belongs to V

⇨ $T^2 + I = 0$

Let $p(x) = x^2 + 1$ and $p(T) = 0$

$m(x) \mid x^2 + 1 \Rightarrow m(x) = x^2 + 1$

Minimal polynomial has two complex (imaginary) roots and imaginary roots always occur in pairs .

Hence , characteristic polynomial is of even degree , hence n is even .

12) Give an example of a linear transformation T:R^2 -> R^2 such that T^2(v) = -v for all v belongs to R^2 .

Solution:

Let T : R^2 -> R^2 given by T (x, y) = (-y, x)

then T^2 (x, y) = T(T(x, y))

= T(-y, x) = (-x, -y) = - (x, y)

⇨ T^2 (v) = -v , for all v belongs to R^2 .

13) Let U, W be sub-sets of V . If U contained in W . Show that A(U) contained in A(W).

Solution:

Let *f* belongs to A(W) then, *f*(w) = 0 for all w belongs to W

⇨ *f*(u) = 0 for all u belongs to U as U contained in W

⇨ *f* belongs to A(U)

Hence , A(U) contained in A(W).

14) Let W_1, W_2 be subspaces of finite dimensional vector space V. Determine $A(W_1 + W_2)$.

Solution :

f belongs to $A(W_1 + W_2)$

⇔ *f(x)* = 0 for all x belongs to $W_1 + W_2$

⇔ *$f(w_1) = 0 = f(w_2)$ for all w_1 belongs to W_1 ,*

W_2 belongs to

W_2

⇔ *f belongs $A(W_1)$ intersection $A(W_2)$*

$A(W_1 + W_2) = A(W_1)$ intersection $A(W_2)$.

15) Let T be a linear operator on V . If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T ?

Solution :

$T^2 = 0 \Rightarrow T^2(v) = 0$ for all v belongs to V

- ⇨ T(T(V)) = 0
- ⇨ T(v) belongs to Ker T , for all v belongs to V
- ⇨ Range T is contained in [Ker T]