RIEMANN INTEGRATION AND SERIES OF FUNCTIONS

MULTIPLE CHOICE QUESTIONS-

1. Find $\int_{-2}^{2} \frac{1}{x+1}dx$.

   (a) Diverges
   (b) 0
   (c) $\frac{1}{2}\ln 3$
   (d) $\frac{8}{9}$
   (e) Ln3

Solution: Function $\frac{1}{x+1}$ has an infinite discontinuity at the point x = −1. Therefore,

$$\int_{-2}^{2} \frac{1}{x+1}dx = \int_{-2}^{-1} \frac{1}{x+1}dx + \int_{-1}^{2} \frac{1}{x+1}dx$$

where each of the integrals is improper. Compute the first integral as follows

$$\int_{-2}^{-1} \frac{1}{x+1}dx = \lim_{t\to -1} \int_{-2}^{t} \frac{1}{x+1}dx = \lim_{t\to -1}[ln|x+1|]_{-2}^{t}$$

$$= \lim_{t\to -1} ln|t+1| - \ln 1 = -\infty$$

Since, $\int_{-2}^{-1} \frac{1}{x+1}$dx diverges, then the initial integral diverges as well.

2. The improper integral $\int_0^{\infty} \frac{dx}{x^2+4}$

   (a) Diverges to ∞.
   (b) Converges to $\frac{\pi}{2}$.
   (c) Converges to $\frac{1}{4}$.
   (d) Diverges to −∞.
   (e) Converges to $\frac{\pi}{4}$.

Solution: $\int_0^{\infty} \frac{dx}{x^2+4}$

$$= \lim_{t\to\infty} \int_0^t \frac{dx}{x^2+4}$$

$=\lim_{t\to\infty} \left[\frac{1}{2} tan^{-1}(\frac{x}{2})\right]_0^t$

$= \lim_{t\to\infty} \frac{1}{2} tan^{-1}(\frac{t}{2})$

$=\frac{1}{2} . \frac{\pi}{2}$

$=\frac{\pi}{4}$

3. Which of the following integrals is/are not improper?Why?

(a) $\int_1^2 \frac{1}{2x-1}$dx

(b) $\int_0^1 \frac{1}{2x-1}$dx

(c) $\int_{\infty}^{\infty} \frac{sinx}{x^2+1}$dx

(d) $\int_1^2 \ln{(x-1)}dx$

Solution:

a) Here both the limits of integration are finite and also the integrand $\frac{1}{2x-1}$ is defined and finite in the given interval of integration [1,2].So, it is a proper integral.

b)Here both the limits are finite but $\frac{1}{2x-1}$ is not defined at 2x-1=0 or x=$\frac{1}{2}$ which lies in the interval [0,1].So,the given integral is Improper of second kind.

c)both the limits of the integration are infinite.So, Here the given integration are infinite. So, the given integral is improper integral of first kind.

d)Here both the limits of integration are finite but the integrand ln(x-1) is not defined at x-1=0 i.e at x=1 , it has infinite discontinuity at x=1.So, the given integral is improper of second kind.

4. Evaluate $\int_{-\infty}^{1} \frac{dx}{(2x-1)^2}$.
   (a) 1
   (b) $\frac{1}{2}$
   (c) $\frac{-1}{2}$
   (d) 0
   (e) Divergent

Solution: $\int_{-\infty}^{1} \frac{dx}{(2x-1)^2}$

$=\lim_{a\to-\infty} \int_{a}^{1} \frac{dx}{(2x-1)^2}$

$=\lim_{a\to-\infty} \frac{1}{2} \int_{a}^{1} \frac{du}{u^2}$

$=\lim_{a\to-\infty} \left(\frac{1}{2}\right) \left[\frac{1}{2x-1}\right]_{a}^{1}$

$=\lim_{a\to-\infty} \left(-\frac{1}{2} + \frac{1}{4a-2}\right)$

$=\mathbf{\frac{-1}{2}}$

5. Find $\int_{-1}^{2} \frac{dx}{x^3}$.

a)-1

b)$\frac{1}{4}$

c)$\frac{-5}{4}$

d)0

e)Divergent

Solution :$\int_{-1}^{2} \frac{dx}{x^3}$.The integrand is not continuous at x=0,so

$$\int_{-1}^{2} \frac{dx}{x^3} = \int_{-1}^{0} \frac{dx}{x^3} + \int_{0}^{2} \frac{dx}{x^3}$$

$$= \lim_{a\to 0^-} \int_{-1}^{a} \frac{dx}{x^3} + \lim_{b\to 0^+} \int_{b}^{2} \frac{dx}{x^3}$$

$$= \lim_{a\to 0^-} \left[\frac{-1}{2x^2}\right]_{-1}^{a} + \lim_{b\to 0^+} \left[\frac{-1}{2x^2}\right]_{b}^{2}$$

$$= \lim_{a\to 0^-} \left(\frac{-1}{2a^2} - \frac{1}{2}\right) + \lim_{b\to 0^+} \left(\frac{-1}{8} + \frac{1}{2b^2}\right)$$

$$= \frac{-1}{0} - \frac{1}{2} + \left(\frac{-1}{8}\right) + \frac{1}{0}$$

$= \frac{1}{0}$ or ∞ (undefined)

DIVERGES.

6. Let f(x) = x for rational x and f(x) = 0 for irrational x. The Upper and Lower Integrals for f over [0, b].

a. U(f)=0=L(f)
b. U(f)=b^2/2=L(f)
c. U(f)=b^2/2 , L(f) = 0
d. None of the above

Solution : f(x) = $\begin{cases} x & , \quad x \in Q \\ 0 & , \quad x \in Q^C \end{cases}$.

∴ , the upper and lower darboux integral over [0,b] are : **U(f)=$\boldsymbol{b^2}$/2 , L(f) = 0**

7. Integral of 1/(x^2+9) over [0, ∞) is

a. Diverges to ∞
b. Converges to π/2
c. Diverges to – ∞
d. Converges to π /6

Solution : Consider,

$$\int_0^\infty \frac{dx}{(x^2+9)} = \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^\infty$$

$$= \frac{1}{3}\left[\frac{\pi}{2} - 0\right]$$

$$= \boldsymbol{\frac{\pi}{6}}$$

8. Let f(x) = 0 for x in [0,1], f(x)= 2 for x in [1,2], f(x)= -1 for x in [2,3]. Then f over [0, 3] is

a. Monotonic
b. Continuous
c. Piecewise Monotonic
d. Piecewise Continuous
e. Riemann Integrable

Solution : f(x) = $\begin{cases} 0 & , \quad x \epsilon [0,1] \\ 2 & , \quad x \epsilon [1,2] \\ -1 & , \quad x \epsilon [2,3] \end{cases}$.

∴ , f is **Riemann integrable** .

9. Find the derivative of g(x)=$\int \sqrt{t^2 + 4t}$ over [1, x] at x = 1?

a. $\sqrt{5}$
b. 2
c. -2
d. None of the above

Solution :g(x)= $\int_1^x \sqrt{t^2 + 4t}$, To Find : g'(1)

g'(x) = $\sqrt{x^2 + 4x}$ (1) - 0

∴ , g'(1) = $\sqrt{1 + 4} = \sqrt{\mathbf{5}}$.

10. Integral of (log x) /x^2 over [3,∞]

a)Converges

b)Diverges

c)Absolute Convergent

d)Conditional Convergent

Solution : Consider ,

$\int_3^\infty \frac{\log x}{x^2}$dx . Let f(x) = $\frac{\log x}{x^2}$

and f(x) is continuous on [3,c] , ∀ c > 3

[Since , log x is continuous for x ϵ $[3, \infty]$ and x^2 is continuous for x ϵ $[3, \infty]$⇒ f(x) = $\frac{\log x}{x^2}$ is continuous for x ϵ $[3, \infty]$**]**

Hence, f(x) is convergent .

SUBJECTIVE QUESTIONS-

11. Is $\int_a^b fg$ converges whenever $\int_a^b f$ and $\int_a^b g$ converges. If no , give an example.

Solution: The answer is **NO**. For example,

$\int_0^1 \frac{1}{\sqrt{x}} dx$ $converges$ but $\int_0^1 (\frac{1}{\sqrt{x}})^2$dx does not .

12. Examine the convergence of $\int_1^\infty \frac{sinx}{x^2}\,dx$.

Solution: we know, $\left|\frac{sinx}{x^2}\right| \leq \frac{1}{x^2}$ and $\frac{1}{x^2}$ is convergent for the interval [1,∞).Hence, by comparison test:

$\int_1^\infty \left|\frac{sinx}{x^2}\right|$ isconvergent.

And hence , by absolute convergence $\Rightarrow \int_1^\infty \frac{sinx}{x^2}$ **is convergent.**

13. Find a function f such that $\int_1^\infty$ fconverges, but $\int_1^\infty \sqrt{\mathrm{f}}$ does not.

Solution: for example, let f(x)=$\frac{1}{x^2}$.Here, is convergent in the given interval of integration but $\sqrt{f}$=$\frac{1}{x}$ is not convergent (since,for $\frac{1}{x^p}$,p>1 converges).

14. Find $\lim\limits_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n$.

Solution:

Divide numerator and denominator by n

$$\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n = \lim_{n\to\infty}\frac{(1+\frac{1}{n})^n}{(1-\frac{1}{n})^n} = \frac{e}{e^{-1}} = e^2$$

15. Determine the values of r for which the integral $\int_1^\infty x^{-r}$ exists and converges.

Solution:

$\int_1^{\infty} x^{-r}$dx = $\int_1^{\infty} \frac{dx}{x^r}$

Consider, $\int_1^{\infty} \frac{dx}{x^r}$ = $\lim_{b\to\infty} \int_1^{b} \frac{dx}{x^r}$ =

$\lim_{b\to\infty} [\frac{x^{-r+1}}{1-r}]_1^b$= $\lim_{b\to\infty} [\frac{b^{1-r}}{1-r} - \frac{1}{1-r}]$

So ,

- ♦ If r=1, the above limit do not exist in $\mathfrak{R}$.
- ♦ If r>1 , 1-r<0 $\Rightarrow \lim_{b\to\infty} b^{1-r}$=0.Therefore, exists and converges.
- ♦ If r<1 ,1-r>0 $\Rightarrow \lim_{b\to\infty} b^{1-r}$=∞.Therefore, does not exists and diverges.

16. Prove or disprove that if $|f|$is integrable on [a,b] , then f is integrable on [a,b] .

Solution :No , the statement is not true as

Suppose f = $\begin{cases} -1 , & x \in Q \\ 1 , & x \in Q^C \end{cases}$. Then ,$|f|$ = 1 , ∀ x ∈$\mathfrak{R}$.

Clearly, |f| is integrable over ℜ , but f is not integrable.

17. Show that $\lim\left(\frac{x}{x+n}\right)$ is uniformly convergent on [0,a], a > 0 .

Solution : Define f : [0,a] $\rightarrow \Re$ *such that* *f(x)* = $\begin{cases} 0 & , \quad x = 0 \\ \frac{a}{a+n} & , \quad x > 0 \end{cases}$.

Consider ,

$\|f_n(x) - f(x)\|$ = sup $\left\{\left|\frac{\frac{x}{x+n} - 0}{\frac{x}{x+n} - \frac{a}{a+n}}\right|\right.$ = sup $\left\{\left|\frac{\frac{x}{x+n}}{\frac{n(x-a)}{(x+n)(a+n)}}\right|\right.$ = 0 .

∴, f is uniformly convergent on [0,a]. → ***HENCE ,PROVED***

18. Show that $f_n(x)$=$\frac{nx}{1+n^2x^2}$, ∀ x ∈ℜ converges pointwise to f(x)=0 on ℜ.

Solution : $\lim_{n\to\infty} \frac{nx}{1+n^2x^2}$ = 0

Define f : ℜ→ℜ by f(x)=0 , ∀ x ∈ℜ , then $f_n(x)$→ f(x) = 0 , ∀ x ∈ℜ.

⇒ (f_n) converges pointwise to f(x) = 0 on ℜ .→ ***HENCE ,PROVED***

19. Suppose f is continuous on [a,b] , f(x) ≥ 0 , ∀ x ∈ [a,b] and $\int_a^b f$ = 0 . Prove that f(x)=0 , ∀ x ∈ [a,b] .

Solution : Suppose g ≠ 0 , ∴, ∃ x ∈ (a,b) such that g(x) ≠ 0.

For $\propto$>0 , suppose g(x)>0

Let [c,d] $\subseteq$ [a,b] such that g(y)> $\propto$; y ϵ [c,d]

$\Rightarrow \int_a^b f \geq \int_c^d g \geq \propto$ (d-c)> 0 $\Rightarrow \int_a^b g$> 0 $\rightarrow$ which is a contradiction .

$\Rightarrow$**g=0** .$\rightarrow$ $\boxed{HENCE\ ,PROVED}$

20. Find radius of convergence of power series

:$\sum_{n=1}^{\infty} \frac{(-4)^n (x+2)^{2n}}{n(n+1)}$.

Solution :Let a_n = $\frac{(-4)^n (x+2)^{2n}}{n(n+1)}$

$\Rightarrow$ROC = $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ = $\lim_{n\to\infty} \left|\frac{\frac{(-4)^n}{n(n+1)}}{\frac{(-4)^{n+1}}{(n+1)(n+2)}}\right|$ = $\lim_{n\to\infty} \left|-\frac{1}{4}\left(1+\frac{2}{n}\right)\right|$

$\Rightarrow \boxed{\boldsymbol{ROC = \frac{1}{4}}}$

21. Prove that $\int_{0^+}^{b} \frac{1}{t^p}$ converges iff p <1 .

Solution :Here , f(t) =$\frac{1}{t^p}$

Let c = $\frac{1}{b-0}$ = $\frac{1}{b}$

Define g on [c,∞[by : g(u) = $\frac{1}{u^2}$ f(0+$\frac{1}{u}$) = $\frac{1}{u^2}.u^p$ = $\frac{1}{u^{2-p}}$

Then ,$\int_{0^+}^{b} f(t)dt = \int_{1/b}^{\infty} \frac{1}{u^{2-p}} du$ [Integral of Type-1]

which converges if 2-p<1

, i.e , p<1.$\rightarrow$ $\boxed{\boldsymbol{HENCE\ ,PROVED}}$

22. Find the pointwise limit of following function

a) $f_n(x) = x/n$

b) $f_n(x) = x^n$, $\forall\, x \in \Re$

Solution :Pointwise limit is determined by $f(x) = \lim f_n(x)$

a. Define $f : \Re \to \Re$ by $f(x)=0$, $\forall\, x \in \Re$
   then , $(f_n) \to f$ pointwise on $\Re$.
b. Define $f : (-1,1] \to \Re$ by $f(x) = \begin{cases} 0 & , \quad -1 < x < 1 \\ 1 & , \quad x = 1 \end{cases}$.
   then , $(f_n) \to f$ pointwise on (-1,1].

23. Check for uniform convergencefor :$f_n(x) = nx/(nx+1)$ on $[0, \infty)$.

Solution : Define $f : [0, \infty) \to \Re$ by $f(x) = 1$, $\forall\, x \in [0, \infty)$
$\Rightarrow (f_n) \to f$ pointwise**on** $[0, \infty)$
Consider ,

$$\|f_n - f\| = \sup\{|f_n - f|\}$$
$$= \sup\{\left|\frac{nx}{(nx+1)} - 1\right|\}$$
$$= \sup \{\left|\frac{1}{(nx+1)}\right| , \forall\, x \in [0, \infty)\}$$
$$= 1 \neq 0$$

$\Rightarrow f_n$ do not converge uniformly to f on $[0, \infty)$.

24. Examine whether $f_n(x) = (x^2+nx)/n$ is uniform convergent or not?

Solution : Define f : $\Re \rightarrow \Re$ by f(x) = x , $\forall$ x $\in \Re$

then , $f_n \rightarrow f$ pointwise on $\Re$.

Consider ,

$$|f_n(x) - f(x)| = \left|\frac{x^2+nx}{n} - x\right| = \left|\frac{x^2}{n}\right|$$

Suppose $n_k = k$, $x_k = k$, $k \in N$

$\Rightarrow |f_n(x) - f(x)| = \left|\frac{k^2}{k}\right| = |k|$ which is unbounded .

$\Rightarrow$ **$f_n(x)$ do not converge uniformly to f(x) , $\forall$ x $\in \Re$.**

25. Show that fn(x) = xe^(-nx) is uniformly convergent on [0, ∞) .

Solution : Define f : $[0, \infty) \rightarrow \Re$ by f(x) = 0 , $\forall$ x $\in \Re$.

then , $f_n \rightarrow f$ pointwise on $[0, \infty)$.

Consider ,

$$|f_n(x) - f(x)| = |xe^{-nx} - 0|$$

Now , $e^{-nx} < \frac{1}{nx} \Rightarrow xe^{-n} < \frac{1}{n}$

$$\Rightarrow |f_n(x) - f(x)| = |xe^{-nx}| < \frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \infty .$$

$\Rightarrow f_n(x)$ converges uniformly to f(x) on $[0, \infty)$.

HENCE , PROVED

26. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)}$.

Solution : Consider ,

$$\int_{-b}^{b} \frac{dx}{(1+x^2)} = 2 \int_0^b \frac{dx}{1+x^2} = 2[\tan^{-1} x]_0^b$$

= $2[\tan^{-1} b - \tan^{-1} 0]$

= $2 \tan^{-1} b$

$\therefore$ **,** $\lim_{n\to\infty} 2 \ \tan^{-1} b$ = $2\left(\frac{\pi}{2}\right)$ = $\boldsymbol{\pi}$ **[FINITE]**

27. Calculate $\lim_{h\to 0}(1/h) \int_3^{3+h} e^{t^2}$ dt .

Solution :$\lim_{h\to 0} \frac{\int_3^{3+h} e^{t^2} dt}{h}$ = $\lim_{h\to 0} \frac{e^{(3+h)^2}.(1)}{1}$ = $\boldsymbol{e^9}$.

28. Show that $\int_0^{0.5} \sin^{-1} x dx$ = $\frac{\pi}{12} - \frac{\sqrt{3}}{2} - 1$

Solution : Let g(x) = sin x on A=[0,$\pi/6$]

As g is increasing (↑) on [0,$\pi/6$] ,$\therefore$ **,** it isone – one.

Let B=g(A) = [sin 0 , sin $\pi/6$] = [0 , 1/2]

$\therefore$,g^{-1} is differentiable on B and

$\int_0^{\pi/6} \sin x \, dx$ + $\int_0^{1/2} \sin^{-1} t dt$ = $\frac{\pi}{6} \sin \frac{\pi}{6}$- 0 sin0

$\therefore$,$\int_0^{1/2} \sin^{-1} t dt$ = $\frac{\pi}{6}\left(\frac{1}{2}\right)$ - $\int_0^{\pi/6} \sin x \, dx$

= $\frac{\pi}{12}$ + $[\cos x]_0^{\frac{\pi}{6}}$ [Fundamental Theorem]

= $\frac{\pi}{12}$ + $\frac{\sqrt{3}}{2} - 1$

HENCE , PROVED

29. Check whether $\int_0^{\infty} e^{-x}$dx is convergent or not.

Solution :Let f(x) be a function defined on (0, ∞) by f(x) = e^{-x}.

Then ∀b >0 , f is continuous on [0,b] .

∴f is integrable on [0,b] .

Thus ,$\int_0^\infty e^{-x}dx$ is an improper integral of **Type-2** for $b > 0$

Consider ,$\int_0^b e^{-x} = \left[-\frac{1}{e^x}\right]_0^b = -\frac{1}{e^b}+1$

$$\therefore, \lim_{b\to\infty} \int_0^b e^{-x}dx = \lim_{b\to\infty}\left(1-\frac{1}{e^b}\right) = 1[\text{FINITE}]$$

∴ It is convergent

30. Calculate $\lim_{x\to\infty}(1/x)\int_0^x e^{t^2}dt$.

Solution :Here , $0 \le t \le x \Rightarrow 0\le t^2 \le x2$

$\Rightarrow e^0 \le e^{t^2} \le e^{x^2} \Rightarrow 1 \le e^{t^2} \le e^{x^2}$

Now, integratingwrt ‘t’ between the limits 0 and x

$$\int_0^x dt \le \int_0^x e^{t^2}dt \le \int_0^x e^{x^2}dt$$

$$\Rightarrow [t]_0^x \le \int_0^x e^{t^2}dt \le e^{x^2}[t]_0^x$$

$\Rightarrow x \le \int_0^x e^{t^2}dt \le xe^{x^2}$

$\Rightarrow 1 \le \frac{1}{x}\int_0^x e^{t^2}dt \le e^{x^2}$

Applying limit :

$\lim_{x\to 0} 1 \le \lim_{x\to 0}\frac{1}{x}\int_0^x e^{t^2}dt \le \lim_{x\to 0} e^{x^2}$

$\Rightarrow 1 \le \lim_{x\to 0}\frac{1}{x}\int_0^x e^{t^2}dt \le 1$

$$\boxed{\lim_{x\to 0}\frac{1}{x}\int_0^x e^{t^2}\,dt = 1}$$ [By Squeeze Theorem]